

On some free boundary problem of the Navier-Stokes equations in the maximal L_p - L_q regularity class

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Abstract

This paper is concerned with the free boundary problem for the Navier Stokes equations without surface tension in the L_p in time and L_q in space setting with $2 < p < \infty$ and $N < q < \infty$. A local in time existence theorem is proved in a uniform $W^{2-1/q}$ domain in the N -dimensional Euclidean space \mathbb{R}^N ($N \geq 2$) under the assumption that weak Dirichlet-Neumann problem is uniquely solvable. Moreover, a global in time existence theorem is proved for small initial data under the assumption that Ω is bounded additionally. This was already proved by Solonnikov [28] by using the continuation argument of local in time solutions which are exponentially stable in the energy level under the assumption that the initial data is orthogonal to the rigid motion. We also use the continuation argument and the same orthogonality for the initial data. But, our argument about the continuation of local in time solutions is based on some decay theorem for the linearized problem, which is a different point than [28].

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1 Introduction

The present paper deals with some local and global in time unique existence theorems of solutions to the Navier-Stokes equations describing the motion of a viscous incompressible fluid flow with free surface without taking surface tension into account. Our problem is formulated in the following. Let Ω be a domain in the N -dimensional Euclidean space \mathbb{R}^N ($N \geq 2$) occupied by a viscous incompressible fluid. We assume that the boundary of Ω consists of two parts S and Γ with $S \cap \Gamma = \emptyset$. We may assume that Γ is an empty set. Let Ω_t and S_t be evolutions of Ω and S with time variable $t > 0$ and we assume that $S_t \cap \Gamma = \emptyset$ for $t \geq 0$. The velocity vector field $\mathbf{v} = \mathbf{v}(x, t) = (v_1(x, t), \dots, v_N(x, t))$ and the pressure $\pi = \pi(x, t)$ for $x = (x_1, \dots, x_N) \in \Omega_t$ satisfy the Navier-Stokes equations

$$(\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v}) - \operatorname{Div} \mathbf{T}(\mathbf{v}, \pi) = 0, \quad \operatorname{div} \mathbf{v} = 0. \quad (1.1)$$

The initial conditions, the boundary conditions on the free boundary S_t and the non-slip conditions on the fixed boundary Γ have the following forms:

$$\begin{aligned} \mathbf{v}|_{t=0} &= \mathbf{v}_0 \quad \text{in } \Omega, \\ \mathbf{T}(\mathbf{v}, \pi) \mathbf{n}_t|_{S_t} &= 0, \quad \mathbf{v}|_{\Gamma} = 0. \end{aligned} \quad (1.2)$$

Here, \mathbf{n}_t is the unit outward normal to S_t . Moreover, $\mathbf{T} = \mathbf{T}(\mathbf{v}, \pi)$ denotes the stress tensor of the form:

$$\mathbf{T}(\mathbf{v}, \pi) = -\pi \mathbf{I} + \mu \mathbf{D}(\mathbf{v}) \quad (1.3)$$

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where μ denotes a positive constant describing the viscosity coefficient, $\mathbf{D}(\mathbf{v})$ the deformation tensor whose (j, k) components are $D_{jk}(\mathbf{v}) = (\partial_j v_k + \partial_k v_j)$ with $\partial_j = \partial/\partial x_j$, and \mathbf{I} the $N \times N$ identity matrix. Finally, for any matrix field \mathbf{K} with components K_{ij} , $i, j = 1, \dots, N$, the quantity $\text{Div } \mathbf{K}$ is an N -vector with i -th component $\sum_{j=1}^N \partial_j K_{ij}$, and also for any vector of functions $\mathbf{u} = (u_1, \dots, u_N)$ we set $\text{div } \mathbf{u} = \sum_{j=1}^N \partial_j u_j$, $\mathbf{u} \cdot \nabla = \sum_{j=1}^N u_j \partial_j$ and $\partial_t \mathbf{u} = (\partial u_1/\partial t, \dots, \partial u_N/\partial t)$.

Aside from the dynamical system (1.1), we impose a further kinematic condition:

$$\partial_t F + (\mathbf{v} \cdot \nabla) F = 0 \quad \text{on } S_t, \quad (1.4)$$

where S_t is defined by $F = F(x, t) = 0$ locally. In other words, S_t is given by

$$S_t = \{x \in \mathbb{R}^N \mid x = \mathbf{x}(\xi, t) \ (\xi \in S)\}, \quad (1.5)$$

where $\mathbf{x} = \mathbf{x}(\xi, t)$ is the solution to the Cauchy problem: $\dot{\mathbf{x}} = d\mathbf{x}/dt = \mathbf{v}(\mathbf{x}, t)$ ($t > 0$) with $\mathbf{x}|_{t=0} = \xi$. This expresses the fact that the free boundary S_t consists of the same particles for all $t > 0$, which do not leave it and are not incident from Ω_t .

The free boundary problem for the Navier-Stokes equations has been studied by many mathematicians in the following two cases:

- (1) The motion of an isolated liquid mass;
- (2) The motion of a viscous incompressible fluid contained in an ocean of infinite content.

In case (1) the initial domain Ω is bounded. A local in time unique existence theorem was proved by Solonnikov [26, 29, 30, 31] in the L_2 Sobolev-Slobodetskii space, by Schweizer [20] in the semigroup setting, by Mogilevskiĭ and Solonnikov [16, 31] in the Hölder spaces with surface tension; and by Solonnikov [28] and Mucha and W. Zajączkowski [18] in the L_p Sobolev-Slobodetskii space and by Shibata and Shimizu [23] in the L_p in time and L_q in space setting without surface tension. A global in time unique existence theorem for small initial velocity was proved by Solonnikov [28] in the L_p Sobolev-Slobodetskii space without surface tension; and by Solonnikov [27] in the L_2 Sobolev-Slobodetskii space and by Padula and Solonnikov [19] in the Hölder spaces under the additional assumption that the initial domain Ω is sufficiently close to a ball with surface tension.

In case (2), the initial domain Ω is a perturbed layer like: $\Omega = \{x \in \mathbb{R}^N \mid -b < X_N < \eta(x'), x' = (x_1, \dots, x_{N-1}) \in \mathbb{R}^{N-1}\}$. A local in time unique existence theorem was proved by Beale [5], Allain [2] and Tani [36] in the L_2 Sobolev-Slobodetskii space with surface tension and by Abels [1] in the L_p Sobolev-Slobodetskii space without surface tension. A global in time unique existence theorem for small initial velocity was proved in the L_2 Sobolev-Slobodetskii space by Beale [6] and Tani and Tanaka [37] with surface tension, and by Sylvester [34] without surface tension. The decay rate was studied by Beale and Nishida [7], Sylvester [35] and Hataya [14].

The purpose of this paper is to prove a local in time unique existence theorem for problem (1.1) and (1.2) under the assumption that the initial domain Ω is a uniform $W_q^{2-1/q}$ ($N < q < \infty$) domain and weak Dirichlet-Neumann problem is uniquely solvable*, which includes the cases (1) and (2) without surface tension. And also, we prove a global in time unique existence theorem for problem (1.1) and (1.2) for a small initial data in the L_p in time and L_q in space setting assuming that Ω is bounded in addition. This was mentioned in Shibata and Shimizu [23], but there was a serious gap in the proof, so that we reprove it in a different approach than [23] in this paper.

To prove a local in time unique existence theorem, the key step is to prove the maximal regularity theorem for the linearized equations given in the following:

$$\begin{aligned} \partial_t \mathbf{u} - \text{Div } \mathbf{T}(\mathbf{u}, \theta) &= \mathbf{f}, \quad \text{div } \mathbf{u} = g = \text{div } \mathbf{g} \quad \text{in } \Omega \times (0, T), \\ \mathbf{T}(\mathbf{u}, \theta) \tilde{\mathbf{n}}|_S &= \mathbf{h}|_S, \quad \mathbf{u}|_\Gamma = 0, \quad \mathbf{u}|_{t=0} = \mathbf{u}_0 \quad \text{in } \Omega \end{aligned} \quad (1.6)$$

with $0 < T \leq \infty$. Here, $\tilde{\mathbf{n}}$ denotes the extension of \mathbf{n} to the whole space \mathbb{R}^N . In fact, as was seen in [13, (5.12)] (cf. also [12, Appendix]), we can define $\tilde{\mathbf{n}}$ on \mathbb{R}^N such that $\tilde{\mathbf{n}}|_S = \mathbf{n}$ and

$$\|f \tilde{\mathbf{n}}\|_{W_q^1(\Omega)} \leq C \|f\|_{W_q^1(\Omega)} \quad (1.7)$$

*These assumptions are exactly stated in Definition 2.1 and Definition 2.2 in the following.

for any $f \in W_q^1(\Omega)$ with some constant C depending on Ω if Ω is a uniform $W_r^{2-1/r}$ domain with $N < r < \infty$.

To prove the maximal regularity theorem, problem (1.6) is reduced locally to the model problems in a neighbourhood of either an interior point or a boundary point by using the localization technique and the partition of unity associated with the domain Ω . The boundary neighbourhood problem (1.6) is transformed to a problem in the half-space $x_N > 0$. By applying the Fourier transform with respect to time and tangential directions, problem (1.6) becomes a system of ordinary differential equations. Solonnikov [25] calculates explicitly the inverse Fourier transform of solutions of such ordinary differential equations and expresses them in the form of potentials in the half-space. Then, he estimates them in suitable norms. Mucha and Zajączkowski [17] directly estimate them using the multiplier theorem of Marinkiewicz and Mikhlin type [15].

On the other hand, Shibata [22] proved the maximal regularity theorem[†] by using the \mathcal{R} -bounded solution operators to the corresponding resolvent problem of the form:

$$\begin{aligned} \lambda \mathbf{v} - \operatorname{Div} \mathbf{T}(\mathbf{v}, \kappa) &= \mathbf{f}, \quad \operatorname{div} \mathbf{v} = g = \operatorname{div} \mathbf{g} \quad \text{in } \Omega, \\ \mathbf{T}(\mathbf{v}, \kappa) \tilde{\mathbf{n}}|_S &= \mathbf{h}|_S, \quad \mathbf{v}|_\Gamma = 0. \end{aligned} \quad (1.8)$$

In fact, according to the theorem in [22], for any $\epsilon \in (0, \pi/2)$ there exist a constant $\lambda_0 \geq 1$ and an operator family $\mathcal{R}(\lambda) \in \operatorname{Hol}(\Sigma_{\epsilon, \lambda_0}, \mathcal{L}(\mathcal{X}_q(\Omega), W_q^2(\Omega)^N))$ such that for any $\mathbf{f} \in L_q(\Omega)^N$, $g \in W_q^1(\Omega)$, $\mathbf{g} \in L_q(\Omega)^N$ and $\mathbf{h} \in W_q^1(\Omega)^N$, problem (1.8) admits a unique solution $\mathbf{v} = \mathcal{R}(\lambda)(\mathbf{f}, \lambda^{1/2}g, \nabla g, \lambda \mathbf{g}, \lambda^{1/2}\mathbf{h}, \nabla \mathbf{h})$ with some pressure term κ , and $(\lambda, \lambda^{1/2}\nabla, \nabla^2)\mathcal{R}(\lambda)$ is \mathcal{R} bounded for $\lambda \in \Sigma_{\epsilon, \lambda_0}$ with value in $\mathcal{L}(\mathcal{X}_q(\Omega), L_q(\Omega)^{\tilde{N}})$. Here, $\tilde{N} = N + N^2 + N^3$, $\Sigma_{\epsilon, \lambda_0} = \{\lambda \in \mathbb{C} \mid |\lambda| \geq \lambda_0, |\arg \lambda| \leq \pi - \epsilon\}$, $\mathcal{X}_q(\Omega) = \{F = (F_1, \dots, F_6) \mid F_1, F_3, F_4, F_5 \in L_q(\Omega)^N, F_2 \in L_q(\Omega), F_6 \in L_q(\Omega)^{N^2}\}$, and F_1, F_2, F_3, F_4, F_5 and F_6 are independent variables corresponding to \mathbf{f} , $\lambda^{1/2}g$, ∇g , $\lambda \mathbf{g}$, $\lambda^{1/2}\mathbf{h}$ and $\nabla \mathbf{h}$, respectively. Moreover, $\operatorname{Hol}(\Sigma_{\epsilon, \lambda_0}, \mathcal{L}(X, Y))$ denotes the set of all $\mathcal{L}(X, Y)$ valued holomorphic functions defined on $\Sigma_{\epsilon, \lambda_0}$ and $\mathcal{L}(X, Y)$ the set of all bounded linear operators from a Banach space X into another Banach space Y . Since the solution \mathbf{u} for (1.6) is given by the Laplace inverse transform of $\mathcal{R}(\lambda)(\mathbf{f}, \lambda^{1/2}g, \nabla g, \lambda \mathbf{g}, \lambda^{1/2}\mathbf{h}, \nabla \mathbf{h})$, the maximal regularity is obtained with help of Weis' operator valued Fourier multiplier theorem [38].

Finally, we introduce some symbols used throughout the paper. For any domain D and $1 \leq q \leq \infty$, $L_q(D)$ and $W_q^m(D)$ denote the usual Lebesgue space and Sobolev space, while $\|\cdot\|_{L_q(D)}$ and $\|\cdot\|_{W_q^m(D)}$ denote their norms, respectively. We set $W_q^0(D) = L_q(D)$. $C_0^\infty(D)$ denotes the set of all $C^\infty(\mathbb{R}^N)$ functions whose supports are compact and contained in D . We set $(f, g)_D = \int_D f(x)g(x) dx$. For any Banach space X and $1 \leq p \leq \infty$, $L_p((a, b), X)$ and $W_p^m((a, b), X)$ denote the usual Lebesgue space and Sobolev space of X -valued functions defined on an interval (a, b) , while $\|\cdot\|_{L_p((a, b), X)}$ and $\|\cdot\|_{W_p^m((a, b), X)}$ denote their norms, respectively. For $0 < \theta < 1$, $B_{q,p}^{2\theta}(D)$ denotes the real interpolation space defined by $B_{q,p}^{2\theta}(D) = (L_q(D), W_q^2(D))_{\theta, p}$ with real interpolation functor $(\cdot, \cdot)_{\theta, p}$, while $\|\cdot\|_{B_{q,p}^{2\theta}(D)}$ denotes its norm. We set $W_q^{2\theta} = B_{q,q}^{2\theta}$. The d -product space of X is defined by $X^d = \{f = (f_1, \dots, f_d) \mid f_i \in X(i = 1, \dots, d)\}$, while its norm is denoted by $\|\cdot\|_X$ instead of $\|\cdot\|_{X^d}$ for the sake of simplicity. \mathbb{N} , \mathbb{R} and \mathbb{C} denote the sets of all natural numbers, real numbers and complex numbers, respectively. We set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For any multi-index $\kappa = (\kappa_1, \dots, \kappa_N) \in \mathbb{N}_0^N$, we write $|\kappa| = \kappa_1 + \dots + \kappa_N$ and $\partial_x^\kappa = \partial_1^{\kappa_1} \dots \partial_N^{\kappa_N}$ with $x = (x_1, \dots, x_N)$ and $\partial_j = \partial/\partial x_j$. For any scalar function f and N -vector of functions \mathbf{g} , we set

$$\begin{aligned} \nabla f &= (\partial_1 f, \dots, \partial_N f), \quad \nabla \mathbf{g} = (\partial_i g_j \mid i, j = 1, \dots, N), \\ \nabla^2 f &= (\partial^\alpha f \mid |\alpha| = 2), \quad \nabla^2 \mathbf{g} = (\partial^\alpha g_i \mid |\alpha| = 2, i = 1, \dots, N). \end{aligned}$$

For $\mathbf{a} = (a_1, \dots, a_N)$ and $\mathbf{b} = (b_1, \dots, b_N) \in \mathbb{R}^N$, we set $\mathbf{a} \cdot \mathbf{b} = \langle \mathbf{a}, \mathbf{b} \rangle = \sum_{j=1}^N a_j b_j$. For scalar functions f, g and N -vectors of functions \mathbf{f}, \mathbf{g} , we set $(f, g)_D = \int_D f(x)g(x) dx$ and $(\mathbf{f}, \mathbf{g})_D = \int_D \mathbf{f}(x) \cdot \mathbf{g}(x) dx$. The letter C denotes generic constants and the constant $C_{a,b,\dots}$ depends on a, b, \dots . The values of constants C and $C_{a,b,\dots}$ may change from line to line.

[†] The maximal regularity theorems are given in Theorem 3.1 and Theorem 3.2 in Sect. 2 in the following.

2 Main Results

In this section, we state our main results. Since Ω_t should be decided, we transfer Ω_t to Ω by the Lagrange transformation as follows: If the velocity field $\mathbf{u}(\xi, t)$ is known as a function of the Lagrange coordinates $\xi \in \Omega$, then the Euler coordinates $x \in \Omega_t$ is written in the form:

$$x = \xi + \int_0^t \mathbf{u}(\xi, s) ds \equiv \mathbf{X}_{\mathbf{u}}(\xi, t),$$

where $\mathbf{u}(\xi, t) = (u_1(\xi, t), \dots, u_N(\xi, t)) = \mathbf{v}(\mathbf{X}_{\mathbf{u}}(\xi, t), t)$. Let A be the Jacobi matrix of the transformation $x = \mathbf{X}_{\mathbf{u}}(\xi, t)$ with elements $a_{ij} = \delta_{ij} + \int_0^t (\partial u_i / \partial \xi_j)(\xi, s) ds$. Since $\det A = 1$ as follows from $\operatorname{div} \mathbf{v} = 0$ in Ω_t , denoting the cofactor matrix of A by \mathcal{A} , we have $\nabla_x = \mathcal{A} \nabla_{\xi}$ with $\nabla_x = {}^T(\partial / \partial x_1, \dots, \partial / \partial x_N)$ and $\nabla_{\xi} = {}^T(\partial / \partial \xi_1, \dots, \partial / \partial \xi_N)$. We can represent \mathcal{A} by $\mathcal{A} = \mathbf{I} + \mathbf{V}_0(\int_0^t \nabla \mathbf{u}(\xi, s) ds)$ with some matrix $\mathbf{V}_0(\mathbf{K})$ of polynomials with respect to $\mathbf{K} = (k_{ij})$ satisfying the condition: $\mathbf{V}_0(0) = 0$, where k_{ij} is a corresponding variable to $\int_0^t (\partial u_i / \partial \xi_j)(\xi, s) ds$. Let \mathbf{n} be the unit outward normal to S , and then by (1.4) we have

$$\mathbf{n}_t = \frac{\mathcal{A} \mathbf{n}}{|\mathcal{A} \mathbf{n}|}. \quad (2.1)$$

We also see that

$$\operatorname{div}_x \mathbf{w} = \operatorname{div}_{\xi} ({}^T \mathcal{A} \hat{\mathbf{w}}) = \operatorname{tr} (\mathcal{A} \nabla_{\xi} \hat{\mathbf{w}}) \quad (2.2)$$

with $\hat{\mathbf{w}}(\xi, t) = \mathbf{w}(\mathbf{X}_{\mathbf{u}}(\xi, t), t)$, where $\operatorname{tr} M$ denotes the trace of any matrix M . Moreover, what $A = (A^{-1})^{-1} = \mathcal{A}^{-1}$ yields that

$$\mathcal{A}^{-1} = \mathbf{I} + \mathbf{V}_1(\int_0^t \nabla \mathbf{u}(\xi, s) ds) \quad (2.3)$$

with some matrix $\mathbf{V}_1(\mathbf{K})$ of polynomials with respect to $\mathbf{K} = (k_{ij})$ satisfying the condition: $\mathbf{V}_1(0) = 0$. Using (2.1), (2.2) and (2.3), and setting $\theta(\xi, t) = \pi(\mathbf{X}_{\mathbf{u}}(\xi, t))$, we have the following Lagrangian description of problem (1.1)-(1.2):

$$\begin{aligned} \partial_t \mathbf{u} - \operatorname{Div} \mathbf{T}(\mathbf{u}, \theta) &= \mathbf{F}(\mathbf{u}), \quad \operatorname{div} \mathbf{u} = G(\mathbf{u}) = \operatorname{div} \mathbf{G}(\mathbf{u}) \quad \text{in } \Omega \times (0, T), \\ \mathbf{T}(\mathbf{u}, \theta) \tilde{\mathbf{n}}|_S &= \mathbf{H}(\mathbf{u}) \tilde{\mathbf{n}}|_S, \quad \mathbf{u}|_{\Gamma} = 0, \quad \mathbf{u}|_{t=0} = \mathbf{v}_0 \quad \text{in } \Omega. \end{aligned} \quad (2.4)$$

Here, $\mathbf{F}(\mathbf{u})$, $g(\mathbf{u})$, $\mathbf{g}(\mathbf{u})$ and $\mathbf{H}(\mathbf{u})$ are nonlinear functions of the forms:

$$\begin{aligned} \mathbf{F}(\mathbf{u}) &= -\mathbf{V}_1(\int_0^t \nabla \mathbf{u} ds) \partial_t \mathbf{u} + \mathbf{V}_2(\int_0^t \nabla \mathbf{u} ds) \nabla^2 \mathbf{u} + \mathbf{V}_3(\int_0^t \nabla \mathbf{u} ds) \int_0^t \nabla^2 \mathbf{u} ds \cdot \nabla \mathbf{u}, \\ G(\mathbf{u}) &= \mathbf{V}_4(\int_0^t \nabla \mathbf{u} ds) \nabla \mathbf{u}, \quad \mathbf{G}(\mathbf{u}) = \mathbf{V}_5(\int_0^t \nabla \mathbf{u} ds) \mathbf{u}, \quad \mathbf{H}(\mathbf{u}) = \mathbf{V}_6(\int_0^t \nabla \mathbf{u} ds) \nabla \mathbf{u}, \end{aligned} \quad (2.5)$$

with some matrices $\mathbf{V}_i(\mathbf{K})$ ($i = 1, \dots, 6$) of polynomials with respect to \mathbf{K} satisfying the conditions:

$$\mathbf{V}_1(0) = 0, \mathbf{V}_2(0) = 0, \mathbf{V}_4(0) = 0, \mathbf{V}_5(0) = 0, \mathbf{V}_6(0) = 0. \quad (2.6)$$

We introduce the definition of uniform $W_r^{2-1/r}$ domain.

Definition 2.1. Let $1 < r < \infty$ and let Ω be a domain in \mathbb{R}^N with boundary $\partial\Omega$. We say that Ω is a uniform $W_r^{2-1/r}$ domain, if there exist positive constants α, β and K such that for any $x_0 = (x_{01}, \dots, x_{0N}) \in \partial\Omega$ there exist a coordinate number j and a $W_r^{2-1/r}$ function $h(x')$ ($x' = (x_1, \dots, \hat{x}_j, \dots, x_N)$) defined on $B'_\alpha(x'_0)$ with $x'_0 = (x_{01}, \dots, \hat{x}_{0j}, \dots, x_{0N})$ and $\|h\|_{W_r^{2-1/r}(B'_\alpha(x'_0))} \leq K$ such that

$$\begin{aligned} \Omega \cap B_\beta(x_0) &= \{x \in \mathbb{R}^N \mid x_j > h(x') \ (x' \in B'_\alpha(x'_0))\} \cap B_\beta(x_0), \\ \partial\Omega \cap B_\beta(x_0) &= \{x \in \mathbb{R}^N \mid x_j = h(x') \ (x' \in B'_\alpha(x'_0))\} \cap B_\beta(x_0). \end{aligned} \quad (2.7)$$

Here, $(x_1, \dots, \hat{x}_j, \dots, x_N) = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_N)$, $B'_\alpha(x'_0) = \{x' \in \mathbb{R}^{N-1} \mid |x' - x'_0| < \alpha\}$ and $B_\beta(x_0) = \{x \in \mathbb{R}^N \mid |x - x_0| < \beta\}$.

${}^{\dagger T} M$ denotes the transposed M .

To prove our local in time unique existence theorem for (2.4) in a uniform $W_q^{2-1/q}$ domain, we need the unique solvability of weak Dirichlet-Neumann problem to treat the divergence condition. But, in general it is not known except for the L_2 framework, so that we have to assume it in this paper. For this purpose, we introduce spaces $W_{q,0}^1(\Omega)$ and $\hat{W}_{q,0}^1(\Omega)$ defined by $\hat{W}_{q,0}^1(\Omega) = \{\theta \in L_{q,\text{loc}}(\Omega) \mid \nabla \theta \in L_q(\Omega)^N, \theta|_S = 0\}$ and $W_{q,0}^1(\Omega) = \{\theta \in W_q^1(\Omega) \mid \theta|_S = 0\}$.

Definition 2.2. Let $1 < q < \infty$ and let $\mathcal{W}_q^1(\Omega)$ be a closed subspace of $\hat{W}_{q,0}^1(\Omega)$ that contains $W_{q,0}^1(\Omega)$. Then, weak Dirichlet-Neumann problem is called uniquely solvable for $\mathcal{W}_q^1(\Omega)$, if the following assertion holds: For any $f \in L_q(\Omega)^N$ there exists a unique $\theta \in \mathcal{W}_q^1(\Omega)$ which satisfies the variational equation:

$$(\nabla \theta, \nabla \varphi)_\Omega = (f, \nabla \varphi)_\Omega \quad \text{for all } \varphi \in \mathcal{W}_{q'}^1(\Omega), \quad (2.8)$$

and the estimate: $\|\nabla \theta\|_{L_q(\Omega)} \leq C_q \|f\|_{L_q(\Omega)}$ for some constant C_q independent of f , θ and φ .

Remark 2.3. (1) $W_q^1(\Omega) + \mathcal{W}_q^1(\Omega) = \{p = p_1 + p_2 \mid p_1 \in W_q^1(\Omega), p_2 \in \mathcal{W}_q^1(\Omega)\}$ is the space for pressures.

(2) When Ω is a bounded domain, a half-space, a perturbed half-space, or a layer domain, weak Dirichlet-Neumann problem is uniquely solvable with $\mathcal{W}_q^1(\Omega) = \hat{W}_{q,0}^1(\Omega)$, while when Ω is an exterior domain, it is uniquely solvable with $\mathcal{W}_q^1(\Omega)$ being the closure of $W_{q,0}^1(\Omega)$ by semi-norm $\|\nabla \cdot\|_{L_q(\Omega)}$. More examples of domains where the unique solvability of weak Dirichlet-Neumann problem holds were given in [21, 22].

To state the compatibility condition for initial data \mathbf{v}_0 , we introduce the solenoidal space $J_q(\Omega)$ defined by $J_q(\Omega) = \{\mathbf{f} \in L_q(\Omega)^N \mid (\mathbf{f}, \nabla \varphi)_\Omega = 0 \text{ for any } \varphi \in \mathcal{W}_{q'}^1(\Omega)\}$. Since $C_0^\infty(\Omega) \subset \mathcal{W}_{q'}^1(\Omega)$, we see that $\text{div } \mathbf{f} = 0$ in Ω provided that $\mathbf{f} \in J_q(\Omega)$. But, the opposite direction does not hold in general. We define $\mathcal{D}_{q,p}(\Omega)$ by $\mathcal{D}_{q,p}(\Omega) = (J_q(\Omega), \mathcal{D}_q(\Omega))_{1-1/p, p}$ with

$$\begin{aligned} \mathcal{D}_q(\Omega) = \{ & \mathbf{f} \in W_q^2(\Omega)^N \mid \mathbf{f} \text{ satisfies the compatibility condition:} \\ & (\mathbf{D}(\mathbf{f})\mathbf{n} - \langle \mathbf{D}(\mathbf{f})\mathbf{n}, \mathbf{n} \rangle \mathbf{n})|_S = 0, \quad \mathbf{f}|_\Gamma = 0\}. \end{aligned} \quad (2.9)$$

From Steiger [32], we know that

$$\mathcal{D}_{q,p}(\Omega) = \begin{cases} \{\mathbf{f} \in B_{q,p}^{2(1-1/p)}(\Omega) \cap J_q(\Omega) \mid \mathbf{f} \text{ satisfies (2.9)}\} & \text{when } 2(1-1/p) > 1+1/q, \\ \{\mathbf{f} \in J_q(\Omega) \cap B_{q,p}^{2(1-1/p)}(\Omega) \mid \mathbf{f}|_\Gamma = 0\} & \text{when } 1/q < 2(1-1/p) < 1+1/q, \\ B_{q,p}^{2(1-1/p)}(\Omega) \cap J_q(\Omega) & \text{when } 2(1-1/p) < 1/q. \end{cases}$$

The following theorem is concerned with local in time unique existence theorem for (2.4).

Theorem 2.4. Let $2 < p < \infty$, $N < q < \infty$ and $R > 0$. Assume that Ω is a uniform $W_q^{2-1/q}$ domain and that weak Dirichlet-Neumann problem is uniquely solvable for $\mathcal{W}_q^1(\Omega)$ and $\mathcal{W}_{q'}^1(\Omega)$ ($q' = q/(q-1)$). Then, there exists a time $T > 0$ depending on R such that for any initial data $\mathbf{v}_0 \in \mathcal{D}_{q,p}(\Omega)$ with $\|\mathbf{v}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)} \leq R$ problem (2.4) admits a unique solution $\mathbf{u} \in L_q((0, T), W_q^2(\Omega)) \cap W_p^1((0, T), L_q(\Omega))$ with some pressure term $\theta \in L_p((0, T), W_q^1(\Omega) + \mathcal{W}_q^1(\Omega))$ possessing the estimate:

$$\|\mathbf{u}\|_{L_p((0, T), W_q^2(\Omega))} + \|\partial_t \mathbf{u}\|_{L_p((0, T), L_q(\Omega))} \leq M_0 R$$

with some positive constant M_0 independent of R and T .

Remark 2.5. (1) Employing the similar argumentation to Strömer [33], we can prove that there exists a positive number $\sigma > 0$ such that the map: $x = \mathbf{X}_\mathbf{u}(\xi, t)$ is diffeomorphism from Ω onto Ω_t , S onto S_t and Γ onto Γ for any $t \in (0, T)$ provided that

$$\int_0^T \|\nabla \mathbf{u}(\cdot, t)\|_{L_\infty(\Omega)} dt \leq \sigma, \quad (2.10)$$

so that from Theorem 2.4 $\mathbf{v}(x, t) = \mathbf{u}(\mathbf{X}_\mathbf{u}^{-1}(x, t), t)$ solves the original free boundary problem (1.1)-(1.2) for small $T > 0$ with some pressure term π , where $\mathbf{X}_\mathbf{u}^{-1}(x, t)$ denotes the inverse map of the correspondence: $x = \mathbf{X}_\mathbf{u}(\xi, t)$.

(2) It is easy to extend Theorem 2.4 to the equation:

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} - \operatorname{Div} \mathbf{T}(\mathbf{v}, \theta) = \mathbf{f}, \quad \operatorname{div} \mathbf{v} = 0 \quad (2.11)$$

instead of (1.1) under similar assumption on \mathbf{f} to Solonnikov [28] and Shibata and Shimizu [23]. But, we only consider the case $\mathbf{f} = 0$ in this paper for simplicity.

Our global in time unique existence theorem is obtained under the assumption that Ω is a bounded domain and the key issue is the orthogonality of the rigid motion. We introduce the rigid space \mathcal{R}_d defined by

$$\mathcal{R}_d = \{Ax + b \mid A : N \times N \text{ anti-symmetric matrix}, b \in \mathbb{R}^N\}. \quad (2.12)$$

We know that \mathbf{u} satisfies $\mathbf{D}(\mathbf{u}) = 0$ if and only if $\mathbf{u} \in \mathcal{R}_d$ (cf. [11]). Let $\{\mathbf{p}_\ell\}_{\ell=1}$ be the orthogonal bases of \mathcal{R}_d , that is $\mathbf{p}_\ell \in \mathcal{R}_d$ ($\ell = 1, \dots, M$) and

$$(\mathbf{p}_\ell, \mathbf{p}_m)_\Omega = \delta_{\ell m} \quad (\ell, m = 1, \dots, M), \quad (2.13)$$

where $\delta_{\ell m}$ is the Kronecker delta symbol such that $\delta_{\ell\ell} = 1$ and $\delta_{\ell m} = 0$ with $\ell \neq m$, M the dimension of \mathcal{R}_d and $(\cdot, \cdot)_\Omega$ the L_2 inner-product on Ω . The following theorem is our global in time unique existence result.

Theorem 2.6. *Let $2 < p < \infty$ and $N < q < \infty$. Assume that Ω is a bounded domain, S and Γ are $W_q^{2-1/q}$ compact hypersurfaces and that $S \neq \emptyset$. Then, there exist numbers $\epsilon > 0$ and $\gamma > 0$ such that for any initial data $\mathbf{v}_0 \in \mathcal{D}_{q,p}(\Omega)$ with $\|\mathbf{v}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)} \leq \epsilon$ that satisfies, in addition, the orthogonality condition:*

$$(\mathbf{v}_0, \mathbf{p}_\ell)_\Omega = 0 \quad (\ell = 1, \dots, M) \quad \text{when } \Gamma = \emptyset, \quad (2.14)$$

problem (2.4) with $T = \infty$ admits a unique solution $\mathbf{u} \in L_p((0, \infty), W_q^2(\Omega)) \cap W_p^1((0, \infty), L_q(\Omega))$ possessing the estimate:

$$\|e^{\gamma t} \mathbf{u}\|_{L_p((0, \infty), W_q^2(\Omega))} + \|e^{\gamma t} \partial_t \mathbf{u}\|_{L_p((0, \infty), L_q(\Omega))} \leq C\epsilon.$$

for some positive constant C independent of ϵ .

3 A proof of a local in time unique existence theorem

In this section, we prove Theorem 2.4. For this purpose, first we state our maximal L_p - L_q regularity theorem obtained by Shibata [22] for the linearized system (1.6). To state our maximal regularity result for (1.6), we introduce some symbols. For any Banach space X with norm $\|\cdot\|_X$, integer $m \geq 0$ and $\gamma_0 > 0$, we set

$$\begin{aligned} W_{p,\gamma_0}^m(I, X) &= \{f : I \rightarrow X \mid e^{-\gamma_0 t} f(t) \in W_p^m(\mathbb{R}_+, X)\} \quad (I = \mathbb{R}_+, \mathbb{R}), \\ W_{p,0,\gamma_0}^m(\mathbb{R}, X) &= \{f : \mathbb{R} \rightarrow X \mid e^{-\gamma_0 t} f(t) \in W_p^m(\mathbb{R}, X), f(t) = 0 \text{ for } t < 0\}, \end{aligned}$$

where $\mathbb{R}_+ = (0, \infty)$. We set $W_{p,\gamma_0}^0 = L_{p,\gamma_0}$ and $W_{p,0,\gamma_0}^0 = L_{p,0,\gamma_0}$. Let \mathcal{L} and \mathcal{L}^{-1} be the Laplace transform and its inverse transform defined by

$$\mathcal{L}[f](\lambda) = \int_{-\infty}^{\infty} e^{-\lambda t} f(t) dt, \quad \mathcal{L}^{-1}[g](t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\lambda t} g(\gamma + i\tau) d\tau$$

with $\lambda = \gamma + i\tau \in \mathbb{C}$. For any real number $s \geq 0$, let $H_{p,\gamma_0}^s(\mathbb{R}, X)$ be the Bessel potential space of order s defined by

$$H_{p,\gamma_0}^s(\mathbb{R}, X) = \{f \in L_{p,\gamma_0}(\mathbb{R}, X) \mid e^{-\gamma t} \Lambda_\gamma^s f \in L_p(\mathbb{R}, X) \text{ for any } \gamma \geq \gamma_0\}$$

with $[\Lambda_\gamma^s f](t) = \mathcal{L}^{-1}[\lambda^s \mathcal{L}[f](\lambda)](t)$. We set $H_{p,0,\gamma_0}^s(\mathbb{R}, X) = \{f \in H_{p,\gamma_0}^s(\mathbb{R}, X) \mid f(t) = 0 \text{ for } t < 0\}$. By using the \mathcal{R} bounded solution operator $\mathcal{R}(\lambda)$ introduced in Sect.1, Shibata [22] proved the following maximal L_p - L_q result for problem (1.6).

Theorem 3.1. *Let $1 < p, q < \infty$, $N < r < \infty$ and $\max(q, q') \leq r$ ($q' = q/(q-1)$). Assume that Ω is a uniform $W_r^{2-1/r}$ domain and that the weak Dirichlet-Neumann problem is uniquely solvable for $\mathcal{W}_q^1(\Omega)$ and $\mathcal{W}_{q'}^1(\Omega)$ ($q' = q/(q-1)$). Then, there exists a positive number γ_0 such that for any initial data $\mathbf{u}_0 \in \mathcal{D}_{q,p}(\Omega)$ and right members \mathbf{f} , g , \mathbf{g} and \mathbf{h} with*

$$\begin{aligned} \mathbf{f} &\in L_{p,0,\gamma_0}(\mathbb{R}, L_q(\Omega)^N), \quad g \in L_{p,0,\gamma_0}(\mathbb{R}, W_q^1(\Omega)) \cap H_{p,0,\gamma_0}^{1/2}(\mathbb{R}, L_q(\Omega)), \\ \mathbf{g} &\in W_{p,0,\gamma_0}^1(\mathbb{R}, L_q(\Omega)^N), \quad \mathbf{h} \in L_{p,0,\gamma_0}(\mathbb{R}, W_q^1(\Omega)^N) \cap H_{p,0,\gamma_0}^{1/2}(\mathbb{R}, L_q(\Omega)^N), \end{aligned}$$

problem (1.6) admits a unique solution $\mathbf{u} \in L_{p,\gamma_0}(\mathbb{R}_+, W_q^2(\Omega)) \cap W_{p,\gamma_0}^1(\mathbb{R}_+, L_q(\Omega)^N)$ with some pressure term $\theta \in L_{p,\gamma_0}(\mathbb{R}_+, W_q^1(\Omega) + \mathcal{W}_q^1(\Omega))$ possessing the estimate:

$$\begin{aligned} &\|e^{-\gamma t} \partial_t \mathbf{u}\|_{L_p(\mathbb{R}_+, L_q(\Omega))} + \|e^{-\gamma t} \mathbf{u}\|_{L_p(\mathbb{R}_+, W_q^2(\Omega))} \leq C \{ \|\mathbf{u}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)} \\ &+ \|e^{-\gamma t} (\mathbf{f}, \partial_t \mathbf{g})\|_{L_p(\mathbb{R}, L_q(\Omega))} + \|e^{-\gamma t} (g, \mathbf{h})\|_{L_p(\mathbb{R}, W_q^1(\Omega))} + \|e^{-\gamma t} \Lambda_\gamma^{1/2} (g, \mathbf{h})\|_{L_p(\mathbb{R}, L_q(\Omega))} \} \end{aligned}$$

for any $\gamma \geq \gamma_0$ with some constant C independent of $\gamma \geq \gamma_0$.

To prove Theorem 2.4, we use the maximal L_p - L_q regularity theorem for problem (1.6) in a finite time interval, which is derived from Theorem 3.1. But, we have to replace the nonlocal operator $\Lambda_\gamma^{1/2}$ with value in $L_q(\Omega)$ by the local operator ∂_t with value in $W_q^{-1}(\Omega)$. For this purpose, first of all, we introduce the extension map $\iota : L_{1,\text{loc}}(\Omega) \rightarrow L_{1,\text{loc}}(\mathbb{R}^N)$ having the following properties:

- (e-1) For any $1 < q < \infty$ and $f \in W_q^1(\Omega)$, $\iota f \in W_q^1(\mathbb{R}^N)$, $\iota f = f$ in Ω and $\|\iota f\|_{W_q^i(\mathbb{R}^N)} \leq C_q \|f\|_{W_q^i(\Omega)}$ for $i = 0, 1$ with some constant C_q depending on q , r and Ω .
- (e-2) For any $1 < q < \infty$ and $f \in W_q^1(\Omega)$, $\|(1 - \Delta)^{-1/2} \iota(\nabla f)\|_{L_q(\mathbb{R}^N)} \leq C_q \|f\|_{L_q(\Omega)}$ with some constant C_q depending on q , r and Ω .

Here, $(1 - \Delta)^{-1/2}$ is the operator defined by $(1 - \Delta)^{-1/2} f = \mathcal{F}^{-1}[(1 + |\xi|^2)^{-1/4} \mathcal{F}f]$ with the help of Fourier transform \mathcal{F} and Fourier inverse transform \mathcal{F}^{-1} which are defined by

$$\mathcal{F}[f](\xi) = \int_{\mathbb{R}^N} e^{-ix \cdot \xi} f(x) dx, \quad \mathcal{F}^{-1}[g](\xi) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{ix \cdot \xi} g(\xi) d\xi.$$

In the following, such extension map ι is fixed. We define $W_q^{-1}(\Omega)$ by

$$W_q^{-1}(\Omega) = \{f \in L_{1,\text{loc}}(\Omega) \mid (1 - \Delta)^{-1/2} \iota f \in L_q(\Omega)\}.$$

As is proved in the appendix below, we have

$$W_{p,0,\gamma_0}^1(\mathbb{R}, W_q^{-1}(\Omega)) \cap L_{p,0,\gamma_0}(\mathbb{R}, W_q^1(\Omega)) \subset H_{p,0,\gamma_0}^{1/2}(\mathbb{R}, L_q(\Omega)), \quad (3.1)$$

$$\|e^{-\gamma t} \Lambda_\gamma^{1/2} f\|_{L_p(\mathbb{R}, L_q(\Omega))} \leq C \{ \|e^{-\gamma t} \partial_t [(1 - \Delta)^{-1/2} (\iota f)]\|_{L_p(\mathbb{R}, L_q(\mathbb{R}^N))} + \|e^{-\gamma t} f\|_{L_p(\mathbb{R}, W_q^1(\Omega))} \} \quad (3.2)$$

for any $\gamma \geq \gamma_0$. Combining Theorem 3.1 with (3.1), we have the following theorem.

Theorem 3.2. *Let $1 < p, q < \infty$, $N < r < \infty$ and $\max(q, q') \leq r$ ($q' = q/(q-1)$). Let T be any positive number. Assume that Ω is a uniform $W_r^{2-1/r}$ domain and that weak Dirichlet-Neumann problem is uniquely solvable for $\mathcal{W}_q^1(\Omega)$ and $\mathcal{W}_{q'}^1(\Omega)$ ($q' = q/(q-1)$). Then, there exists a positive number γ_0 such that for any initial data $\mathbf{u}_0 \in \mathcal{D}_{q,p}(\Omega)$ and any right members \mathbf{f} , g , \mathbf{g} and \mathbf{h} with*

$$\begin{aligned} \mathbf{f} &\in L_p((0, T), L_q(\Omega)^N), \quad g \in L_p((0, T), W_q^1(\Omega)) \cap W_p^1((0, T), W_q^{-1}(\Omega)), \\ \mathbf{g} &\in W_p^1((0, T), L_q(\Omega)^N), \quad \mathbf{h} \in L_p((0, T), W_q^1(\Omega)^N) \cap W_p^1((0, T), W_q^{-1}(\Omega)^N), \end{aligned}$$

satisfying the conditions: $g|_{t=0} = 0$, $\mathbf{g}|_{t=0} = 0$ and $\mathbf{h}|_{t=0} = 0$, problem (1.6) admits a unique solution $\mathbf{u} \in L_p((0, T), W_q^2(\Omega)^N) \cap W_p^1((0, T), L_p(\Omega)^N)$ with pressure term $\theta \in L_p((0, T), W_q^1(\Omega) + \mathcal{W}_q^1(\Omega))$ possessing the estimate:

$$\begin{aligned} &\|\mathbf{u}\|_{L_p((0,t), W_q^2(\Omega))} + \|\partial_t \mathbf{u}\|_{L_p((0,t), L_q(\Omega))} \leq C e^{\gamma t} \{ \|\mathbf{u}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)} \\ &+ \|(\mathbf{f}, \partial_t \mathbf{g})\|_{L_p((0,t), L_q(\Omega))} + \|(g, \mathbf{h})\|_{L_p((0,t), W_q^1(\Omega))} + \|\partial_t [(1 - \Delta)^{-1/2} (\iota g, \iota \mathbf{h})]\|_{L_p((0,t), L_q(\mathbb{R}^N))} \} \end{aligned}$$

for any $t \in (0, T]$ and $\gamma \geq \gamma_0$ with some constant C independent of $\gamma \geq \gamma_0$ and $t \in (0, T]$.

Proof. Let t be any number with $0 < t \leq T$. Given $f(\cdot, s)$ defined for $s \geq 0$, $f_0(\cdot, s)$ denotes the zero extension of f to $s < 0$, that is $f_0(\cdot, s) = f(\cdot, s)$ for $s \geq 0$ and $f_0(\cdot, s) = 0$ for $s < 0$. Let $E_t f$ be the extension of f defined by

$$E_t f = \begin{cases} f_0(\cdot, s) & \text{for } s \leq t, \\ f_0(\cdot, 2t - s) & \text{for } s \geq t. \end{cases} \quad (3.3)$$

Note that $E_t f$ vanishes for $s \notin [0, 2t]$. Moreover, if $f|_{s=0}$, then

$$\partial_s E_t f = \begin{cases} \partial_s f(\cdot, s) & \text{for } s \leq t, \\ -(\partial_s f)(\cdot, 2t - s) & \text{for } s \geq t, \\ 0 & \text{for } s \notin [0, 2t]. \end{cases} \quad (3.4)$$

Let $\mathbf{u}^t = \mathbf{v}(\cdot, s)$ and $\theta^t = \kappa(\cdot, s)$ be solutions to the equations:

$$\begin{aligned} \partial_s \mathbf{v} - \operatorname{Div} \mathbf{T}(\mathbf{v}, \kappa) &= E_t \mathbf{f}, \quad \operatorname{div} \mathbf{v} = E_t g = \operatorname{div} (E_t \mathbf{g}) \quad \text{in } \Omega \times (0, \infty), \\ \mathbf{T}(\mathbf{v}, \kappa) \tilde{\mathbf{n}}|_S &= E_t \mathbf{h}|_S, \quad \mathbf{v}|_\Gamma = 0, \quad \mathbf{v}|_{t=0} = \mathbf{u}_0 \quad \text{in } \Omega. \end{aligned} \quad (3.5)$$

Since $E_{t_1} f = E_{t_2} f$ for $0 < t_1, t_2 \leq T$, by the uniqueness of solutions yields that $\mathbf{u}^{t_1}(\cdot, s) = \mathbf{u}^{t_2}(\cdot, s)$ for $s \in [0, t_1]$ with $0 < t_1 < t_2 \leq T$. By Theorem 3.1

$$\begin{aligned} \|e^{-\gamma s} \partial_s \mathbf{u}^t\|_{L_p(\mathbb{R}, L_q(\Omega))} + \|e^{-\gamma s} \mathbf{u}^t\|_{L_p(\mathbb{R}, W_q^2(\Omega))} &\leq C\{\|\mathbf{u}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)} + \|e^{-\gamma s} (E_t \mathbf{f}, \partial_s (E_t \mathbf{g}))\|_{L_p(\mathbb{R}, L_q(\Omega))} \\ &\quad + \|e^{-\gamma s} (E_t g, E_t \mathbf{h})\|_{L_p(\mathbb{R}, W_q^1(\Omega))} + \|e^{-\gamma s} \Lambda_\gamma^{1/2} (E_t g, E_t \mathbf{h})\|_{L_p(\mathbb{R}, L_q(\Omega))}\}. \end{aligned} \quad (3.6)$$

Noting (3.4), we see easily that

$$\begin{aligned} \|e^{-\gamma s} (E_t \mathbf{f}, \partial_s (E_t \mathbf{g}))\|_{L_p(\mathbb{R}, L_q(\Omega))} &\leq 2\|e^{-\gamma s} (\mathbf{f}, \partial_s \mathbf{g})\|_{L_p((0,t), L_q(\Omega))}, \\ \|e^{-\gamma s} (E_t g, E_t \mathbf{h})\|_{L_p(\mathbb{R}, W_q^1(\Omega))} &\leq 2\|e^{-\gamma s} (g, \mathbf{h})\|_{L_p((0,t), W_q^1(\Omega))}. \end{aligned} \quad (3.7)$$

Moreover, by (3.2) and (3.4), we have

$$\begin{aligned} &\|e^{-\gamma s} \Lambda_\gamma^{1/2} (E_t g, E_t \mathbf{h})\|_{L_p(\mathbb{R}, L_q(\Omega))} \\ &\leq C\{\|e^{-\gamma s} \partial_s [(1 - \Delta)^{-1/2} (\iota g, \iota \mathbf{h})]\|_{L_p((0,t), L_q(\mathbb{R}^N))} + \|e^{-\gamma s} (g, \mathbf{h})\|_{L_p((0,t), W_q^1(\Omega))}\}. \end{aligned} \quad (3.8)$$

Setting $\mathbf{u} = \mathbf{u}^T$ and $\theta = \theta^T$, noting that $\mathbf{u}(\cdot, s) = \mathbf{u}^t(\cdot, s)$ for $0 < s < t$ and combining (3.6), (3.7) and (3.8), we complete the proof of Theorem 3.2. \square

A Proof of Theorem 2.4 In the following, we assume that $2 < p < \infty$ and $N < q < \infty$, that Ω is a uniform $W_q^{2-1/q}$ domain in \mathbb{R}^N ($N \geq 2$), and that weak Dirichlet-Neumann problem is uniquely solvable for $\mathcal{W}_q^1(\Omega)$ and $\mathcal{W}_{q'}^1(\Omega)$ ($q' = q/(q-1)$). By Sobolev's imbedding theorem we have

$$W_q^1(\Omega) \subset L_\infty(\Omega), \quad \left\| \prod_{j=1}^m f_j \right\|_{W_q^1(\Omega)} \leq C \prod_{j=1}^m \|f_j\|_{W_q^1(\Omega)}. \quad (3.9)$$

Let T and L be any positive numbers and we define a space $\mathcal{I}_{L,T}$ by

$$\mathcal{I}_{L,T} = \{\mathbf{v} \in L_p((0, T), W_q^2(\Omega)) \cap W_p^1((0, T), L_q(\Omega)) \mid \mathbf{v}|_{t=0} = \mathbf{v}_0 \quad \text{in } \Omega, \quad \mathbb{I}_{\mathbf{v}}(0, T) \leq L\}, \quad (3.10)$$

where we have set $\mathbb{I}_{\mathbf{v}}(0, T) = \|\mathbf{v}\|_{L_p((0,T), W_q^2(\Omega))} + \|\partial_t \mathbf{v}\|_{L_p((0,T), L_q(\Omega))}$. Given $\mathbf{w} \in \mathcal{I}_{L,T}$, let \mathbf{v} and ω be solutions to problem:

$$\begin{aligned} \partial_t \mathbf{v} - \operatorname{Div} \mathbf{T}(\mathbf{v}, \omega) &= \mathbf{F}(\mathbf{w}), \quad \operatorname{div} \mathbf{v} = G(\mathbf{w}) = \operatorname{div} \mathbf{G}(\mathbf{w}) \quad \text{in } \Omega \times (0, T), \\ \mathbf{T}(\mathbf{v}, \omega) \tilde{\mathbf{n}}|_S &= \mathbf{H}(\mathbf{w}) \tilde{\mathbf{n}}|_S, \quad \mathbf{v}|_\Gamma = 0, \quad \mathbf{v}|_{t=0} = \mathbf{v}_0 \quad \text{in } \Omega. \end{aligned} \quad (3.11)$$

First, we estimate the right-hand sides of (3.11). By (3.9) and Hölder's inequality we have

$$\sup_{t \in (0, T)} \left\| \int_0^t \nabla \mathbf{w}(\cdot, s) ds \right\|_{L_\infty(\Omega)} \leq M_1 T^{1/p'} L, \quad \sup_{t \in (0, T)} \left\| \int_0^t \nabla \mathbf{w}(\cdot, s) ds \right\|_{W_q^1(\Omega)} \leq C T^{1/p'} L. \quad (3.12)$$

with $p' = p/(p-1)$. Here and in the following, C denotes a generic constant independent of T and R and we use the letters M_i to denote some special constants independent of T and L . The value of C may change from line to line. To treat nonlinear functions with respect to $\int_0^t \nabla \mathbf{w}(\cdot, s) ds$, we choose T so small that $M_1 T^{1/p'} L \leq 1$ in (3.12), so that

$$\sup_{t \in (0, T)} \left\| \int_0^t \nabla \mathbf{w}(\cdot, s) ds \right\|_{L_\infty(\Omega)} \leq 1. \quad (3.13)$$

By (3.12), (3.13), (3.9) and (2.6), we have

$$\sup_{t \in (0, T)} \left\| \mathbf{V}_i \left(\int_0^t \nabla \mathbf{w}(\cdot, s) ds \right) \right\|_{W_q^1(\Omega)} \leq C T^{1/p'} L, \quad \sup_{t \in (0, T)} \left\| \nabla \mathbf{W} \left(\int_0^t \nabla \mathbf{w}(\cdot, s) ds \right) \right\|_{L_q(\Omega)} \leq C T^{1/p'} L \quad (3.14)$$

where $i = 1, 2, 4, 5$ and 6 , and $\mathbf{W} = \mathbf{W}(\mathbf{K})$ is any matrix of polynomials with respect to \mathbf{K} . By (2.5), (1.7), (3.9), (3.12), (3.13) and (3.14), we have

$$\|\mathbf{F}(\mathbf{w})\|_{L_p((0, T), L_q(\Omega))} \leq C L^2 T^{1/p'}, \quad \|(G(\mathbf{w}), \mathbf{H}(\mathbf{w})\tilde{\mathbf{n}})\|_{L_p((0, T), W_q^1(\Omega))} \leq C L^2 T^{1/p'}. \quad (3.15)$$

To obtain

$$\sup_{t \in (0, T)} \|\mathbf{w}(\cdot, t)\|_{B_{q,p}^{2(1-1/p)}(\Omega)} \leq C(\mathbb{I}_{\mathbf{w}}(0, T) + e^{\gamma T} \|\mathbf{v}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)}), \quad (3.16)$$

we use the embedding relation:

$$L_p((0, \infty), X_1) \cap W_p^1((0, \infty), X_0) \subset BUC(J, [X_0, X_1]_{1-1/p, p}) \quad (3.17)$$

for any two Banach spaces X_0 and X_1 such that X_1 is dense in X_0 and $1 < p < \infty$ (cf. [3]). In fact, let E_t be the extension operator defined in the proof of Theorem 3.2 and let \mathbf{Z} and Π be solutions to problem:

$$\begin{aligned} \partial_t \mathbf{Z} - \operatorname{Div} \mathbf{T}(\mathbf{Z}, \Pi) &= 0, \quad \operatorname{div} \mathbf{Z} = 0 \quad \text{in } \Omega \times (0, \infty), \\ \mathbf{T}(\mathbf{Z}, \Pi)\tilde{\mathbf{n}}|_S &= 0, \quad \mathbf{Z}|_\Gamma = 0, \quad \mathbf{Z}|_{t=0} = \mathbf{v}_0 \quad \text{in } \Omega. \end{aligned} \quad (3.18)$$

By Theorem 3.1 (1), we know the unique existence of (\mathbf{Z}, Π) possessing the estimate:

$$\|e^{-\gamma t} \partial_t \mathbf{Z}\|_{L_p(\mathbb{R}_+, L_q(\Omega))} + \|e^{-\gamma t} \mathbf{Z}\|_{L_p(\mathbb{R}_+, W_q^2(\Omega))} \leq C \|\mathbf{v}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)} \quad (\gamma \geq \gamma_0) \quad (3.19)$$

for some constants γ_0 and C , where C is independent of $\gamma \geq \gamma_0$. We choose γ so large and fix it in the following. Set $\mathbf{z} = \mathbf{w} - \mathbf{Z}$. Since $\mathbf{z}|_{t=0} = 0$, by (3.3) and (3.4) we have

$$\mathbb{I}_{E_T \mathbf{z}}(0, \infty) \leq C \mathbb{I}_{\mathbf{z}}(0, T) \leq C(\mathbb{I}_{\mathbf{w}}(0, T) + e^{\gamma T} \mathbb{I}_{e^{-\gamma t} \mathbf{Z}}(0, \infty)).$$

Thus, noting that $\mathbf{w} = \mathbf{Z} + E_T \mathbf{z}$ for $t \in (0, T)$ and using (3.17), we have

$$\begin{aligned} \sup_{t \in (0, T)} \|\mathbf{w}(\cdot, t)\|_{B_{q,p}^{2(1-1/p)}(\Omega)} &\leq \sup_{t \in (0, \infty)} \|E_T \mathbf{z}(\cdot, t)\|_{B_{q,p}^{2(1-1/p)}(\Omega)} + e^{\gamma T} \sup_{t \in (0, \infty)} \|e^{-\gamma t} \mathbf{Z}(\cdot, t)\|_{B_{q,p}^{2(1-1/p)}(\Omega)} \\ &\leq C(\mathbb{I}_{E_T \mathbf{z}}(0, \infty) + e^{\gamma T} \mathbb{I}_{e^{-\gamma t} \mathbf{Z}}(0, \infty)) \leq C(\mathbb{I}_{\mathbf{w}}(0, T) + e^{\gamma T} \mathbb{I}_{e^{-\gamma t} \mathbf{Z}}(0, \infty)), \end{aligned}$$

which combined with (3.19) furnishes (3.16).

Since $B_{q,p}^{2(1-1/p)}(\Omega) \subset W_q^1(\Omega)$ as follows from the assumption: $2 < p < \infty$, by (3.16) and (3.13) we have

$$\begin{aligned} \sup_{t \in (0, T)} \|\mathbf{w}(\cdot, t)\|_{W_q^1(\Omega)} &\leq C(L + e^{\gamma T} \|\mathbf{v}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)}), \\ \sup_{t \in (0, T)} \left\| \partial_t \mathbf{W} \left(\int_0^t \nabla \mathbf{w}(\cdot, s) ds \right) \right\|_{L_q(\Omega)} &\leq C(L + e^{\gamma T} \|\mathbf{v}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)}). \end{aligned} \quad (3.20)$$

Writing $\partial_t \mathbf{G}(\mathbf{w}) = \{\partial_t \mathbf{V}_5(\int_0^t \nabla \mathbf{w} ds)\} \mathbf{w} + \mathbf{V}_5(\int_0^t \nabla \mathbf{w} ds) \partial_t \mathbf{w}$ and using (2.6), (3.9), (3.16), (3.20) and (3.14), we have

$$\|\partial_t \mathbf{G}(\mathbf{w})\|_{L_p((0,T), L_q(\Omega))} \leq C\{L^2 T^{1/p'} + (L + e^{\gamma T} \|\mathbf{v}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)})^2 T^{1/p}\}. \quad (3.21)$$

To continue our estimate, we prepare the following lemma.

Lemma 3.3. *Let $1 < p < \infty$, $N < q, r < \infty$ and let Ω be a uniform $W_r^{2-1/r}$ domain. Let ι be the extension map satisfying the properties (e-1) and (e-2). Then,*

$$\begin{aligned} & \|\partial_t[(1 - \Delta)^{-1/2} \iota((\nabla f)g)]\|_{L_p((0,T), L_q(\mathbb{R}^N))} \\ & \leq C\left\{\left(\int_0^T (\|\partial_t f(\cdot, t)\|_{L_q(\Omega)} \|g(\cdot, t)\|_{W_q^1(\Omega)})^p dt\right)^{1/p} + \left(\int_0^T (\|\nabla f(\cdot, t)\|_{L_q(\Omega)} \|\partial_t g(\cdot, t)\|_{L_q(\Omega)})^p dt\right)^{1/p}\right\}. \end{aligned}$$

Proof. To prove the lemma, we use an inequality:

$$\|(1 - \Delta)^{-1/2} \iota(fg)\|_{L_q(\mathbb{R}^N)} \leq C\|f\|_{L_q(\Omega)} \|g\|_{L_q(\Omega)} \quad (3.22)$$

provided that $N < q < \infty$, which follows from the following observation: For any $\varphi \in C_0^\infty(\mathbb{R}^N)$ by Hölder's inequality and (e-1) we have

$$|((1 - \Delta)^{-1/2} \iota(fg), \varphi)_{\mathbb{R}^N}| = |(\iota(fg), (1 - \Delta)^{-1/2} \varphi)_{\mathbb{R}^N}| \leq C\|f\|_{L_q(\Omega)} \|g\|_{L_q(\Omega)} \|(1 - \Delta)^{-1/2} \varphi\|_{L_s(\mathbb{R}^N)},$$

where s is an index such that $2/q + 1/s = 1$. Since $N(1/q' - 1/s) = N/q < 1$, by Sobolev's imbedding theorem we have $\|(1 - \Delta)^{-1/2} \varphi\|_{L_s(\mathbb{R}^N)} \leq C\|\varphi\|_{L_{q'}(\mathbb{R}^N)}$, which furnishes (3.22). Since

$$\partial_t[(1 - \Delta)^{-1/2} \iota((\nabla f)g)] = (1 - \Delta)^{-1/2} \iota[\nabla\{(\partial_t f)g\}] - (1 - \Delta)^{-1/2} \iota[(\partial_t f)(\nabla g)] + (1 - \Delta)^{-1/2} \iota[(\nabla f) \partial_t g],$$

by (3.22), (3.9) and (e-2) we have Lemma 3.3. \square

Applying Lemma 3.3 to $G(\mathbf{w})$ and $\mathbf{H}(\mathbf{w})\tilde{\mathbf{n}}$ with $f = \mathbf{w}$, $g = \mathbf{V}_4(\int_0^t \nabla(\mathbf{w}) ds)$ and $f = \mathbf{w}$, $g = \mathbf{V}_6(\int_0^t \nabla(\mathbf{w}) ds)\tilde{\mathbf{n}}$, respectively, and using (3.14), (3.16) and (3.20), we have

$$\begin{aligned} & \|\partial_t[(1 - \Delta)^{-1/2} (\iota G(\mathbf{w}), \iota(\mathbf{H}(\mathbf{w})\tilde{\mathbf{n}}))]\|_{L_p((0,T), L_q(\mathbb{R}^N))} \\ & \leq C\{L^2 T^{1/p'} + (L + e^{\gamma T} \|\mathbf{v}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)})^2 T^{1/p}\}. \end{aligned} \quad (3.23)$$

Thus, applying Theorem 3.2 to problem (3.11) and using (3.15), (3.21) and (3.23), we have

$$\mathbb{I}_{\mathbf{v}}(0, T) \leq M_2 \|\mathbf{v}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)} + M_3(L^2 T^{1/p'} + (L + e^{\gamma T} \|\mathbf{v}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)})^2 T^{1/p}). \quad (3.24)$$

Let R be a number such that $\|\mathbf{v}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)} \leq R$ and set $L = (M_2 + 1)R$. Choosing $T > 0$ so small that

$$M_3(L^2 T^{1/p'} + (L + e^{\gamma T} R)^2 T^{1/p}) \leq 1,$$

by (3.24) we have $\mathbb{I}_{\mathbf{v}}(0, T) \leq L$, so that $\mathbf{v} \in \mathcal{I}_{L,T}$. If we define a map Φ by $\Phi(\mathbf{w}) = \mathbf{v}$, then Φ is a map from $\mathcal{I}_{L,T}$ into itself.

Next, we show the contractility of the map Φ on $\mathcal{I}_{L,T}$. Let $\mathbf{w}_i \in \mathcal{I}_{L,T}$ and set $\mathbf{v}_i = \Phi(\mathbf{w}_i)$ ($i = 1, 2$). Setting $\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2$, we have

$$\begin{aligned} \partial_t \mathbf{v} - \text{Div } \mathbf{T}(\mathbf{v}, \omega) &= \mathbf{f}(\mathbf{w}_1, \mathbf{w}_2), \quad \text{div } \mathbf{v} = g(\mathbf{w}_1, \mathbf{w}_2) = \text{div } \mathbf{g}(\mathbf{w}_1, \mathbf{w}_2) \quad \text{in } \Omega \times (0, T), \\ \mathbf{T}(\mathbf{v}, \omega)\tilde{\mathbf{n}}|_S &= \mathbf{h}(\mathbf{w}_1, \mathbf{w}_2)|_S, \quad \mathbf{v}|_\Gamma = 0, \quad \mathbf{v}|_{t=0} \quad \text{in } \Omega \end{aligned} \quad (3.25)$$

with some pressure term ω , where we have set

$$\begin{aligned} \mathbf{f}(\mathbf{w}_1, \mathbf{w}_2) &= \mathbf{F}(\mathbf{w}_1) - \mathbf{F}(\mathbf{w}_2), \quad g(\mathbf{w}_1, \mathbf{w}_2) = G(\mathbf{w}_1) - G(\mathbf{w}_2), \\ \mathbf{g}(\mathbf{w}_1, \mathbf{w}_2) &= \mathbf{G}(\mathbf{w}_1) - \mathbf{G}(\mathbf{w}_2), \quad \mathbf{h}(\mathbf{w}_1, \mathbf{w}_2) = (\mathbf{H}(\mathbf{w}_1) - \mathbf{H}(\mathbf{w}_2))\tilde{\mathbf{n}} \end{aligned}$$

By Theorem 3.2, we have

$$\mathbb{I}_{\mathbf{v}_1 - \mathbf{v}_2}(0, T) \leq M_4 \mathbb{J}(\mathbf{w}_1, \mathbf{w}_2)(T) \quad (3.26)$$

for some constant M_4 independent of T and R with

$$\begin{aligned} \mathbb{J}(\mathbf{w}_1, \mathbf{w}_2) &= \|(\mathbf{f}(\mathbf{w}_1, \mathbf{w}_2), \partial_t \mathbf{g}(\mathbf{w}_1, \mathbf{w}_2))\|_{L_p(0, T), L_q(\Omega)} + \|(g(\mathbf{w}_1, \mathbf{w}_2), \mathbf{h}(\mathbf{w}_1, \mathbf{w}_2))\|_{L_p((0, T), W_q^1(\Omega))} \\ &\quad + \|\partial_t[(1 - \Delta)^{-1/2}(\iota g(\mathbf{w}_1, \mathbf{w}_2), \iota \mathbf{h}(\mathbf{w}_1, \mathbf{w}_2))]\|_{L_p((0, T), L_q(\mathbb{R}^N))}. \end{aligned}$$

We estimate each terms in the right-hand side of (3.25). Recalling that $\|\mathbf{v}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)} \leq R$ and $L = (M_2 + 1)R$, by (3.12), (3.13), (3.14), (3.16) and (3.20) we have

$$\begin{aligned} \sup_{t \in (0, T)} \left\| \int_0^t \nabla \mathbf{w}_i(\cdot, s) ds \right\|_{L_\infty(\Omega)} &\leq 1, & \sup_{t \in (0, T)} \left\| \int_0^t \nabla \mathbf{w}_i(\cdot, s) ds \right\|_{W_q^1(\Omega)} &\leq CRT^{1/p'}, \\ \sup_{t \in (0, T)} \left\| \mathbf{V}_j \left(\int_0^t \nabla \mathbf{w}_i(\cdot, s) ds \right) \right\|_{W_q^1(\Omega)} &\leq CRT^{1/p'}, & \sup_{t \in (0, T)} \left\| \nabla \mathbf{W} \left(\int_0^t \nabla \mathbf{w}_i(\cdot, s) ds \right) \right\|_{W_q^1(\Omega)} &\leq CRT^{1/p'}, \\ \sup_{t \in (0, T)} \left\| \mathbf{w}_i(\cdot, t) \right\|_{W_q^1(\Omega)} &\leq CRe^{\gamma T}, & \sup_{t \in (0, T)} \left\| \partial_t \mathbf{W} \left(\int_0^t \nabla \mathbf{w}_i(\cdot, s) ds \right) \right\|_{L_q(\Omega)} &\leq CRe^{\gamma T}, \end{aligned}$$

where $i = 1, 2$ and $j = 1, 2, 4, 5, 6$. Thus, we have

$$\begin{aligned} \sup_{t \in (0, T)} \left\| \int_0^t \nabla \mathbf{w}_1(\cdot, s) ds - \int_0^t \nabla \mathbf{w}_2(\cdot, s) ds \right\|_{W_q^1(\Omega)} &\leq CT^{1/p'} \mathbb{I}_{\mathbf{w}_1 - \mathbf{w}_2}(0, T), \\ \sup_{t \in (0, T)} \left\| \mathbf{W} \left(\int_0^t \nabla \mathbf{w}_1(\cdot, s) ds \right) - \mathbf{W} \left(\int_0^t \nabla \mathbf{w}_2(\cdot, s) ds \right) \right\|_{W_q^1(\Omega)} &\leq C(RT^{1/p'} + 1)T^{1/p'} \mathbb{I}_{\mathbf{w}_1 - \mathbf{w}_2}(0, T). \end{aligned}$$

Since $(\mathbf{w}_1 - \mathbf{w}_2)|_{t=0} = 0$, employing the similar argumentation to that in the proof of (3.16), we have

$$\begin{aligned} \sup_{t \in (0, T)} \left\| \mathbf{w}_1(\cdot, t) - \mathbf{w}_2(\cdot, t) \right\|_{W_q^1(\Omega)} &\leq C \mathbb{I}_{\mathbf{w}_1 - \mathbf{w}_2}(0, T), \\ \sup_{t \in (0, T)} \left\| \partial_t \left\{ \mathbf{W} \left(\int_0^t \nabla \mathbf{w}_1(\cdot, s) ds \right) - \mathbf{W} \left(\int_0^t \nabla \mathbf{w}_2(\cdot, s) ds \right) \right\} \right\|_{W_q^1(\Omega)} &\leq C(1 + Re^{\gamma T} T^{1/p'}) \mathbb{I}_{\mathbf{w}_1 - \mathbf{w}_2}(0, T). \end{aligned}$$

Using above estimates and Lemma 3.3, we have

$$\mathbb{J}(\mathbf{w}_1, \mathbf{w}_2)(T) \leq M_5 \mathcal{C}(R, T) \mathbb{I}_{\mathbf{w}_1 - \mathbf{w}_2}(0, T) \quad (3.27)$$

for some constant M_5 independent of R and T with

$$\begin{aligned} \mathcal{C}(R, T) &= RT^{1/p'} + (RT^{1/p'})^2 + (RT^{1/p'})^3 + RT^{1/p} \\ &\quad + e^{\gamma T} \{ (RT^{1/p})(RT^{1/p'}) + RT^{1/p} + RT^{1/p'} + (RT^{1/p'})^2 \} + e^{2\gamma T} (RT^{1/p'}) (RT^{1/p}). \end{aligned}$$

Combining (3.27) with (3.26) furnishes that

$$\mathbb{I}_{\Phi(\mathbf{w}_1) - \Phi(\mathbf{w}_2)}(0, T) \leq M_4 M_5 \mathcal{C}(R, T) \mathbb{I}_{\mathbf{w}_1 - \mathbf{w}_2}(0, T). \quad (3.28)$$

Choosing T smaller in such a way that $M_4 M_5 \mathcal{C}(R, T) \leq 1/2$, we have Φ is a contraction map on $\mathcal{I}_{L, T}$. Thus, the Banach fixed point theorem tells us that Φ has a unique fixed point \mathbf{u} in $\mathcal{I}_{L, T}$ satisfying the equations (2.4).

Finally, we prove the uniqueness. Given two $\mathbf{v}_i \in \mathcal{I}_{L, T}$ ($i = 1, 2$) both of which satisfy the equations (2.4) with the same initial data $\mathbf{v}_0 \in B_{q,p}^{2(1-1/p)}(\Omega)$, employing the same argument as in proving (3.28) and replacing \mathbf{w}_i by \mathbf{v}_i , we have $\mathbb{I}_{\mathbf{v}_1 - \mathbf{v}_2}(0, T) \leq M_4 M_5 \mathcal{C}(R, T) \mathbb{I}_{\mathbf{v}_1 - \mathbf{v}_2}(0, T)$. Since T has been chosen in such a way that $M_4 M_5 \mathcal{C}(R, T) \leq 1/2$, we have $\mathbb{I}_{\mathbf{v}_1 - \mathbf{v}_2}(0, T) \leq \frac{1}{2} \mathbb{I}_{\mathbf{v}_1 - \mathbf{v}_2}(0, T)$, which implies that $\mathbf{v}_1 = \mathbf{v}_2$. This completes the proof of Theorem 2.4.

4 Some decay properties of solutions to problem (1.6)

In this section, we discuss exponential stability of solutions to problem (1.6) assuming that Ω is bounded in addition. Let $\mathcal{R}(\lambda)$ be the \mathcal{R} bounded solution operator for problem (1.8) introduced in Sect. 1. If we consider the time shifted equation of (1.6):

$$\begin{aligned} \partial_t \mathbf{v} + \lambda_1 \mathbf{v} - \operatorname{Div} \mathbf{T}(\mathbf{v}, \hat{\theta}) &= \mathbf{f}, \quad \operatorname{div} \mathbf{v} = g = \operatorname{div} \mathbf{g} \quad \text{in } \Omega \times (0, \infty), \\ \mathbf{T}(\mathbf{v}, \hat{\theta}) \tilde{\mathbf{n}}|_S &= \mathbf{h}|_S, \quad \mathbf{v}|_\Gamma = 0, \quad \mathbf{v}|_{t=0} = \mathbf{u}_0 \quad \text{in } \Omega, \end{aligned} \quad (4.1)$$

a solution \mathbf{v} is represented by using $\mathcal{R}(\lambda + \lambda_1)$, so that we have the following theorem concerning the exponential stability of solutions to (4.1).

Theorem 4.1. *Let $1 < p, q < \infty$, $N < r < \infty$ and $\max(q, q') \leq r$ ($q' = q/(q-1)$). Assume that Ω is a uniform $W_r^{2-1/r}$ domain and that weak Dirichlet-Neumann problem is uniquely solvable for $\mathcal{W}_q^1(\Omega)$ and $\mathcal{W}_q^1(\Omega)$ ($q' = q/(q-1)$). Then, there exists a $\lambda_1 > \lambda_0$ such that problem (4.1) admits a unique solution $(\mathbf{v}, \hat{\theta})$ with*

$$\mathbf{v} \in L_p(\mathbb{R}_+, W_q^2(\Omega)) \cap W_p^1(\mathbb{R}_+, L_q(\Omega)), \quad \hat{\theta} \in L_p(\mathbb{R}_+, W_q^1(\Omega) + \mathcal{W}_q^1(\Omega))$$

possessing the estimate:

$$\begin{aligned} &\|e^{\gamma t} \partial_t \mathbf{v}\|_{L_p(\mathbb{R}_+, L_q(\Omega))} + \|e^{\gamma t} \mathbf{v}\|_{L_p(\mathbb{R}_+, W_q^2(\Omega))} \\ &\leq C(\|\mathbf{u}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)} + \|e^{\gamma t}(\mathbf{f}, \tilde{\Lambda}_\gamma^{1/2} g, \partial_t \mathbf{g}, \tilde{\Lambda}_\gamma^{1/2} \mathbf{h})\|_{L_p(\mathbb{R}, L_q(\Omega))} + \|e^{\gamma t}(g, \mathbf{h})\|_{L_p(\mathbb{R}, W_q^1(\Omega))}) \end{aligned}$$

for any $\gamma \leq \lambda_0$ with some constants C independent of $\gamma \leq \lambda_0$, provided that $\mathbf{u}_0 \in \mathcal{D}_{q,p}(\Omega)$,

$$\begin{aligned} e^{\gamma t} \mathbf{f} &\in L_p(\mathbb{R}, L_q(\Omega)^N), \quad e^{\gamma t} g \in L_p(\mathbb{R}, W_q^1(\Omega)), \quad e^{\gamma t} \tilde{\Lambda}_\gamma^{1/2} g \in L_p(\mathbb{R}, L_q(\Omega)), \\ e^{\gamma t} \partial_t \mathbf{g} &\in L_p(\mathbb{R}, L_q(\Omega)^N), \quad e^{\gamma t} \mathbf{h} \in L_p(\mathbb{R}, W_q^1(\Omega)^N), \quad e^{\gamma t} \tilde{\Lambda}_\gamma^{1/2} \mathbf{h} \in L_p(\mathbb{R}, L_q(\Omega)^N), \end{aligned} \quad (4.2)$$

and \mathbf{f} , g , \mathbf{g} and \mathbf{h} vanish for $t < 0$. Here, we have defined $\tilde{\Lambda}_\gamma^{1/2} f$ by

$$\tilde{\Lambda}_\gamma^{1/2} f = \mathcal{L}_\lambda^{-1}[(\lambda + \lambda_1)^{1/2} \mathcal{L}[f](\lambda)] \quad \text{with } \lambda = -\gamma + i\tau. \quad (4.3)$$

Since the \mathcal{R} boundedness implies the usual boundedness of operators, we also see that for any $\lambda \in \Sigma_{\epsilon, \lambda_0}$, $\mathbf{f} \in L_q(\Omega)^N$, $g \in W_q^1(\Omega)$, $\mathbf{g} \in L_q(\Omega)^N$ and $\mathbf{h} \in W_q^1(\Omega)^N$, a unique solution \mathbf{v} of problem (1.8) possesses the generalized resolvent estimate:

$$\|(|\lambda| \mathbf{v}, |\lambda|^{1/2} \nabla \mathbf{v}, \nabla^2 \mathbf{v})\|_{L_q(\Omega)} \leq C\|(\mathbf{f}, |\lambda|^{1/2} g, \nabla g, |\lambda| \mathbf{g}, |\lambda|^{1/2} \mathbf{h}, \nabla \mathbf{h})\|_{L_q(\Omega)} \quad (4.4)$$

with some constant C depending on ϵ and λ_0 . Especially, we see the existence of a continuous semigroup $\{\mathbb{T}(t)\}_{t \geq 0}$ associated with problem (1.6), which is analytic.

To prove a global in time unique existence theorem for (2.4), we need the exponential stability of solutions to (1.6), so that from now on, we assume that Ω is bounded in addition. In this case, weak Dirichlet-Neumann problem is uniquely solvable for any exponent $q \in (1, \infty)$ with $\mathcal{W}_q^1(\Omega) = \tilde{W}_{q,0}^1(\Omega)$ and $J_q(\Omega) = \{\mathbf{f} \in L_q(\Omega)^N \mid \operatorname{div} \mathbf{f} = 0, \mathbf{n}_\Gamma \cdot \mathbf{f}|_\Gamma = 0\}$, where \mathbf{n}_Γ is the unit outer normal to Γ . When Ω is bounded, the uniqueness of solutions to problem (1.8) holds when $\Gamma \neq \emptyset$ up to $\lambda = 0$. When $\Gamma = \emptyset$, if we restrict the space of solutions to the quotient space $W_q^2(\Omega)/\mathcal{R}_d$, then we also have the uniqueness of solutions to (1.8). Namely, if $\mathbf{u} \in W_q^2(\Omega)$ satisfies the equations (1.8) with $\mathbf{f} = 0$, $g = 0$, $\mathbf{g} = 0$ and $\mathbf{h} = 0$ and if \mathbf{u} satisfies the orthogonal condition: $(\mathbf{u}, \mathbf{p}_\ell)_\Omega = 0$ for $\ell = 1, \dots, M$, then $\mathbf{u} = 0$ up to $\lambda = 0$. Moreover, if $\mathbf{f} \in L_q(\Omega)^N$ and $\mathbf{g} \in W_q^1(\Omega)^N$ satisfy the condition: $(\mathbf{f}, \mathbf{p}_\ell)_\Omega + \langle \mathbf{h}, \mathbf{p}_\ell \rangle_S = 0$, then a solution \mathbf{u} to problem (1.8) also satisfies $(\mathbf{u}, \mathbf{p}_\ell)_\Omega = 0$ whenever $\lambda \neq 0$. Here, $\langle f, g \rangle_S = \int_\Gamma f(x)g(x) d\sigma$, $d\sigma$ being the surface element of S . Using these facts and applying a homotopic argument, we see that $\{\mathbb{T}(t)\}_{t \geq 0}$ is exponentially stable. Namely, we have the following theorem which was already proved in Shibata and Shimizu [23] in the case of $\Gamma = \emptyset$.

Theorem 4.2. *Let $1 < q < \infty$, $N < r < \infty$ and $\max(q, q') \leq r$ ($q' = q/(q-1)$). Assume that Ω is a uniform $W_r^{2-1/r}$ domain and that Ω is bounded in addition. Then, there exists a continuous semigroup $\{\mathbb{T}(t)\}_{t \geq 0}$ on $J_q(\Omega)$ associated with problem (1.6) such that $\mathbf{u} = \mathbb{T}(t)\mathbf{u}_0$ with some pressure term θ solves problem (1.6) with $\mathbf{f} = 0$, $g = 0$, $\mathbf{g} = 0$ and $\mathbf{h} = 0$. Moreover, $\{\mathbb{T}(t)\}_{t \geq 0}$ is analytic and exponentially stable, that is*

$$\|\mathbb{T}(t)\mathbf{u}_0\|_{W_q^\ell(\Omega)} \leq C(1 + t^{-\ell/2})e^{-\gamma t}\|\mathbf{u}_0\|_{L_q(\Omega)} \quad \text{for any } t > 0 \text{ and } \ell = 0, 1, 2 \quad (4.5)$$

with some positive constants C and γ provided that $\mathbf{u}_0 \in J_q(\Omega)$ when $\Gamma \neq \emptyset$ and $\mathbf{u}_0 \in J_q(\Omega)$ satisfying the orthogonal condition: $(\mathbf{u}_0, \mathbf{p}_\ell)_\Omega = 0$ for $\ell = 1, \dots, M$ when $\Gamma = \emptyset$. Here, $W_q^0(\Omega) = L_q(\Omega)$.

By Theorem 4.2, we have the following Corollary which was proved in Shibata and Shimizu [23] in the case of $\Gamma = \emptyset$ under the assumption that the boundary of Ω is a $C^{1,1}$ hypersurface.

Corollary 4.3. *Let $1 < q < \infty$, $N < r < \infty$ and $\max(q, q') \leq r$ ($q' = q/(q-1)$). Assume that Ω is a uniform $W_r^{2-1/r}$ domain and that Ω is bounded in addition. Then, there exists a positive constant γ_0 such that problem (1.6) with $\mathbf{f} = 0$, $g = 0$, $\mathbf{g} = 0$ and $\mathbf{h} = 0$ admits unique solutions \mathbf{u} and θ with*

$$\mathbf{u} \in L_p(\mathbb{R}_+, W_q^2(\Omega)^N) \cap W_p^1(\mathbb{R}_+, L_q(\Omega)^N), \quad \theta \in L_p(\mathbb{R}_+, W_q^1(\Omega) + \hat{W}_{q,0}^1(\Omega))$$

possessing the estimate:

$$\|e^{\gamma t} \partial_t \mathbf{u}\|_{L_p(\mathbb{R}_+, L_q(\Omega))} + \|e^{\gamma t} \mathbf{u}\|_{L_p(\mathbb{R}_+, W_q^2(\Omega))} \leq C \|\mathbf{u}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)} \quad \text{for any } \gamma \leq \gamma_0$$

with some positive constants C independent of $\gamma \leq \gamma_0$ provided that $\mathbf{u}_0 \in \mathcal{D}_{q,p}(\Omega)$ when $\Gamma \neq \emptyset$ and $\mathbf{u}_0 \in \mathcal{D}_{q,p}(\Omega)$ and \mathbf{u}_0 satisfies the orthogonal condition: $(\mathbf{u}_0, \mathbf{p}_\ell)_\Omega = 0$ for $\ell = 1, \dots, M$ when $\Gamma = \emptyset$.

Under the preparations mentioned above, we show the following theorem about the exponential stability of solutions to (1.6).

Theorem 4.4. *Let $1 < p, q < \infty$, $N < r < \infty$ and $\max(q, q') \leq r$ ($q' = q/(q-1)$). Assume that Ω is a uniform $W_r^{2-1/r}$ domain and that Ω is bounded in addition. Then, there exists a positive constant γ_0 such that the following assertion holds: Let $\mathbf{u}_0 \in \mathcal{D}_{q,p}(\Omega)$, and let right members \mathbf{f} , g , \mathbf{g} , and \mathbf{h} for (1.6) satisfy the decay condition (4.2) and vanish for $t < 0$, then problem (1.6) with $T = \infty$ admits a unique solution $\mathbf{u} \in L_p(\mathbb{R}_+, W_q^2(\Omega)^N) \cap W_p^1(\mathbb{R}_+, L_q(\Omega)^N)$ with some pressure term $\theta \in L_p(\mathbb{R}_+, W_q^1(\Omega) + \hat{W}_{q,0}^1(\Omega))$ possessing the estimate:*

$$\|e^{\gamma t} \partial_t \mathbf{u}\|_{L_p((0,T), L_q(\Omega))} + \|e^{\gamma t} \mathbf{u}\|_{L_p((0,T), W_q^2(\Omega))} \leq C \{ \mathbf{J}_{p,q} + \delta(\Gamma) \sum_{\ell=1}^M \left(\int_0^T |e^{\gamma t} (\mathbf{u}(\cdot, t), \mathbf{p}_\ell)_\Omega|^p dt \right)^{1/p} \} \quad (4.6)$$

for any $T > 0$ and $\gamma \leq \gamma_0$ with some constant C independent of T . Here, $\delta(\Gamma)$ is a constant defined by $\delta(\Gamma) = 1$ when $\Gamma = \emptyset$ and $\delta(\Gamma) = 0$ if $\Gamma \neq \emptyset$, and we have set

$$\mathbf{J}_{p,q} = \|\mathbf{u}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)} + \|e^{\gamma t} (\mathbf{f}, \tilde{\Lambda}_\gamma^{1/2} g, \partial_s \mathbf{g}, \tilde{\Lambda}_\gamma^{1/2} \mathbf{h})\|_{L_p(\mathbb{R}, L_q(\Omega))} + \|e^{\gamma t} (g, \mathbf{h})\|_{L_p(\mathbb{R}, W_q^1(\Omega))}.$$

Proof. We look for a solution \mathbf{u} of the form $\mathbf{u} = \mathbf{v} + \mathbf{w}$, where \mathbf{v} and \mathbf{w} are a solution to (4.1) and a solution to problem:

$$\begin{aligned} \partial_t \mathbf{w} - \operatorname{Div} \mathbf{T}(\mathbf{w}, \tilde{\theta}) &= \lambda_1 \mathbf{v}, \quad \operatorname{div} \mathbf{w} = 0 \quad \text{in } \Omega \times (0, \infty), \\ \mathbf{T}(\mathbf{w}, \tilde{\theta}) \tilde{\mathbf{n}}|_S &= 0, \quad \mathbf{w}|_\Gamma = 0, \quad \mathbf{w}|_{t=0} = 0 \quad \text{in } \Omega \end{aligned} \quad (4.7)$$

with some pressure term $\tilde{\theta} \in L_p(\mathbb{R}_+, W_q^1(\Omega) + \hat{W}_{q,0}^1(\Omega))$, respectively. By Theorem 4.1

$$\|e^{\gamma t} \partial_t \mathbf{v}\|_{L_p(\mathbb{R}_+, L_q(\Omega))} + \|e^{\gamma t} \mathbf{v}\|_{L_p(\mathbb{R}_+, W_q^2(\Omega))} \leq C \mathbf{J}_{p,q}. \quad (4.8)$$

If $\Gamma \neq \emptyset$, setting

$$\mathbf{w}(\cdot, t) = \int_0^t \mathbb{T}(t-s)(\lambda_1 \mathbf{v}(\cdot, s)) ds,$$

by Duhamel's principle we see that \mathbf{w} satisfies (4.7). Moreover, setting

$$\mathbf{L}_{q,\mathbf{a}}(t) = \|\mathbf{a}(\cdot, t)\|_{W_q^2(\Omega)} + \|\partial_t \mathbf{a}(\cdot, t)\|_{L_q(\Omega)},$$

by Theorem 4.2 and Hölder's inequality we have

$$\mathbf{L}_{q,\mathbf{w}}(t) \leq C \int_0^t e^{-\gamma_0(t-s)} \mathbf{L}_{q,\mathbf{v}}(s) ds \leq C(\gamma_0 p')^{-1/p'} \left(\int_0^t e^{-\gamma_0 p(t-s)} \mathbf{L}_{q,\mathbf{v}}(s)^p ds \right)^{1/p}$$

with some $\gamma_0 > 0$ for some positive constant C independent of $t > 0$, where $p' = p/(p-1)$. Thus, for $\gamma < \gamma_0$ we have

$$\begin{aligned} \int_0^T (e^{\gamma t} \mathbf{L}_{q,\mathbf{w}}(t))^p dt &\leq C(\gamma_0 p')^{-p/p'} \int_0^T \mathbf{L}_{q,\mathbf{v}}(s)^p \left(\int_s^T e^{-\gamma_0 p(t-s)} e^{\gamma p t} dt \right) ds \\ &= C(\gamma_0 p')^{-p/p'} \int_0^T (e^{\gamma s} \mathbf{L}_{q,\mathbf{v}}(s))^p \left(\int_s^T e^{-(\gamma_0 - \gamma)p(t-s)} dt \right) ds \\ &\leq C(\gamma_0 p')^{-p/p'} ((\gamma_0 - \gamma)p)^{-1} \int_0^T (e^{\gamma s} \mathbf{L}_{q,\mathbf{v}}(s))^p ds, \end{aligned}$$

which combined with (4.8) furnishes (4.6) with $\delta(\Gamma) = 0$.

Next, we consider the case of $\Gamma = \emptyset$. Setting $\mathbf{z}(x, t) = \lambda_1 \mathbf{v}(x, t) - \sum_{\ell=1}^M (\lambda_1 \mathbf{v}(\cdot, t), \mathbf{p}_\ell)_\Omega \mathbf{p}_\ell(x)$, we have $(\mathbf{z}(t), \mathbf{p}_\ell)_\Omega = 0$ for $\ell = 1, \dots, M$ and $t > 0$. Writing $\tilde{\mathbf{w}}(t) = \int_0^t \mathbb{T}(t-s) \mathbf{z}(s) ds$, by Duhamel's principle, we see that $\tilde{\mathbf{w}}$ satisfies (4.7) replacing $\lambda_1 \mathbf{v}$ by \mathbf{z} . Moreover, by Theorem 4.2 and Hölder's inequality

$$\mathbf{L}_{q,\tilde{\mathbf{w}}}(t) \leq C \int_0^t e^{-\gamma_0(t-s)} \mathbf{L}_{q,\mathbf{z}}(s) ds.$$

Thus, by (4.8) we have

$$\int_0^T (e^{\gamma t} \mathbf{L}_{q,\tilde{\mathbf{w}}}(t))^p dt \leq C_{\gamma_0, \gamma, p} \int_0^T (e^{\gamma t} \mathbf{L}_{q,\mathbf{z}}(t))^p dt \leq C_{\gamma_0, \gamma, p} (\mathbf{J}_{p,q})^p. \quad (4.9)$$

Setting $\mathbf{u} = \mathbf{v} + \mathbf{w}$ with $\mathbf{w} = \tilde{\mathbf{w}} + \sum_{\ell=1}^M \int_0^t (\lambda_1 \mathbf{v}(\cdot, s), \mathbf{p}_\ell)_\Omega \mathbf{p}_\ell$, we see that \mathbf{u} satisfies (1.6) with some pressure term $\theta \in L_p(\mathbb{R}_+, W_q^1(\Omega) + \dot{W}_{q,0}^1(\Omega))$, because $\mathbf{D}(\mathbf{p}_\ell) = 0$ and $\operatorname{div} \mathbf{p}_\ell = 0$. Moreover, we have $\mathbf{D}(\mathbf{u}) = \mathbf{D}(\mathbf{v}) + \mathbf{D}(\tilde{\mathbf{w}})$, so that by (4.8) and (4.9) we have

$$\int_0^T \|\mathbf{D}(\mathbf{u}(\cdot, t))\|_{L_q(\Omega)}^p dt \leq C(\mathbf{J}_{p,q})^p \quad (4.10)$$

for any $T > 0$ with some constant C independent of T . Since

$$\|\mathbf{a}\|_{W_q^1(\Omega)} \leq C(\|\mathbf{D}(\mathbf{a})\|_{L_q(\Omega)} + \sum_{\ell=1}^M |(\mathbf{a}, \mathbf{p}_\ell)_\Omega|)$$

for any $\mathbf{a} \in W_q^1(\Omega)^N$ as follows from the usual contradiction argument (cf. Duvaut and Lions [11]), by (4.10)

$$\int_0^T (e^{\gamma t} \|\mathbf{u}(\cdot, t)\|_{W_q^1(\Omega)})^p dt \leq C\{(\mathbf{J}_{p,q})^p + \sum_{\ell=1}^M \int_0^T e^{\gamma p t} |(\mathbf{u}(\cdot, t), \mathbf{p}_\ell)_\Omega|^p dt\}. \quad (4.11)$$

In addition, by (4.4) with $\lambda = \lambda_0 + 1$, we have

$$\|\mathbf{u}(t)\|_{W_q^2(\Omega)} \leq C\{\|\partial_t \mathbf{u}(t)\|_{L_q(\Omega)} + \|\mathbf{u}(t)\|_{L_q(\Omega)} + \|(\mathbf{f}(t), \mathbf{g}(t))\|_{L_q(\Omega)} + \|(g(t), \mathbf{h}(t))\|_{W_q^1(\Omega)}\}. \quad (4.12)$$

Since $\partial_t \mathbf{u} = \partial_t \mathbf{v} + \partial_t \tilde{\mathbf{w}} + \lambda_1 \sum_{\ell=1}^M (\mathbf{v}(t), \mathbf{p}_\ell)_\Omega \mathbf{p}_\ell$, by (4.8), (4.9), (4.11) and (4.12) we have (4.6), which completes the proof of Theorem 4.4. \square

Finally, we prove the following theorem with help of Theorem 4.4.

Theorem 4.5. *Let $1 < p, q < \infty$, $N < r < \infty$ and $\max(q, q') \leq r$ ($q' = q/(q-1)$). Let T be any positive number. Assume that Ω is a uniform $W_r^{2-1/r}$ domain and that Ω is bounded in addition. Then, there exists a positive constant γ_0 such that for any $\mathbf{u}_0 \in \mathcal{D}_{p,q}(\Omega)$ and right members \mathbf{f} , g , \mathbf{g} and \mathbf{h} with*

$$\begin{aligned} \mathbf{f} &\in L_p((0, T), L_q(\Omega)^N), \quad g \in L_p((0, T), W_q^1(\Omega)) \cap W_p^1((0, T), W_q^{-1}(\Omega)), \\ \mathbf{g} &\in W_p^1((0, T), L_q(\Omega)^N), \quad \mathbf{h} \in L_p((0, T), W_q^1(\Omega)^N) \cap W_p^1((0, T), W_q^{-1}(\Omega)^N), \end{aligned}$$

satisfying the condition: $g|_{t=0} = 0$, $\mathbf{g}|_{t=0} = 0$ and $\mathbf{h}|_{t=0} = 0$, problem (1.6) admits unique solutions \mathbf{u} and θ with

$$\mathbf{u} \in L_p((0, T), W_q^2(\Omega)^N) \cap W_p^1((0, T), L_q(\Omega)^N), \quad \theta \in L_p((0, T), W_q^1(\Omega) + \hat{W}_{q,0}^1(\Omega))$$

possessing the estimate:

$$\begin{aligned} &\|e^{\gamma t} \partial_t \mathbf{u}\|_{L_p((0,t), L_q(\Omega))} + \|e^{\gamma t} \mathbf{u}\|_{L_p((0,t), W_q^2(\Omega))} \\ &\leq C e^{2\gamma_0} \{\|\mathbf{u}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)} + \delta(\Gamma) \sum_{\ell=1}^M \left(\int_0^t |e^{\gamma s} (\mathbf{u}(\cdot, s), \mathbf{p}_\ell)_\Omega|^p ds \right)^{1/p} + \|e^{\gamma s} (\mathbf{f}, \partial_t \mathbf{g})\|_{L_p((0,t), L_q(\Omega))} \\ &\quad + \|e^{\gamma s} (g, \mathbf{h})\|_{L_p((0,T), W_q^1(\Omega))} + \|e^{\gamma s} \partial_s [(1-\Delta)^{-1/2}(\iota g, \iota \mathbf{h})]\|_{L_p((0,t), L_q(\mathbb{R}^N))}\} \end{aligned}$$

for any $t \in (0, T]$ and $0 < \gamma \leq \gamma_0$ with some constant C independent of T and γ . Here, $\delta(\Gamma)$ is the same number as in Theorem 4.4.

Proof. Let E_t be the same operator as in the proof of Theorem 3.2. Let $\phi(s)$ be a function in $C^\infty(\mathbb{R})$ such that $\phi(s) = 1$ for $s \leq 0$ and $\phi(s) = 0$ for $s \geq 1$ and set $\phi_t(s) = \phi(s-t)$. Obviously, $\phi_t \in C^\infty(\mathbb{R})$, $\phi_t(s) = 1$ for $s \leq t$ and $\phi_t(s) = 0$ for $s \geq t+1$. Let $\mathbf{u}^t = \mathbf{v}$ and $\theta^t = \omega$ be solutions to the equations:

$$\begin{aligned} \partial_s \mathbf{v} - \operatorname{Div} \mathbf{T}(\mathbf{v}, \omega) &= \phi_t E_t \mathbf{f}, \quad \operatorname{div} \mathbf{v} = \phi_t E_t g = \operatorname{div} (\phi_t E_t \mathbf{g}) \quad \text{in } \Omega \times (0, \infty), \\ \mathbf{T}(\mathbf{v}, \omega) \tilde{\mathbf{n}}|_S &= \phi_t E_t \mathbf{h}|_S, \quad \mathbf{v}|_\Gamma = 0, \quad \mathbf{v}|_{s=0} = \mathbf{u}_0 \quad \text{in } \Omega. \end{aligned} \quad (4.13)$$

Since $(\phi_t E_t f)(\cdot, s) = f(\cdot, s)$ for $s \in [0, T]$, \mathbf{u}^t and θ^t solve problem (1.6) for $s \in (0, t)$. And, by the uniqueness of solutions, $\mathbf{u}^{t_1}(\cdot, s) = \mathbf{u}^{t_2}(\cdot, s)$ for $s \in [0, t_1]$ when $0 < t_1 < t_2 \leq T$. By Theorem 4.4,

$$\begin{aligned} &\|e^{\gamma s} \mathbf{u}^t\|_{L_p((0,t), W_q^2(\Omega))} + \|e^{\gamma s} \partial_s \mathbf{u}^t\|_{L_p((0,t), L_q(\Omega))} \leq C \{\|\mathbf{u}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)} \\ &\quad + \delta(\Gamma) \sum_{\ell=1}^M \left(\int_0^t |e^{\gamma s} (\mathbf{u}(\cdot, s), \mathbf{p}_\ell)_\Omega|^p ds \right)^{1/p} + \|e^{\gamma s} (\phi_t E_t \mathbf{f}, \partial_s (\phi_t E_t \mathbf{g}))\|_{L_p(\mathbb{R}, L_q(\Omega))} \\ &\quad + \|e^{\gamma s} (\phi_t E_t g, \phi_t E_t \mathbf{h})\|_{L_p(\mathbb{R}, W_q^1(\Omega))} + \|e^{\gamma s} (\tilde{\Lambda}_\gamma^{1/2}(\phi_t E_t g), \tilde{\Lambda}_\gamma^{1/2}(\phi_t E_t \mathbf{h}))\|_{L_p(\mathbb{R}, L_q(\Omega))}\}. \end{aligned} \quad (4.14)$$

Let X be $L_q(\Omega)$ or $W_q^1(\Omega)$. Using the change of variable: $2t - s = r$, we have

$$\begin{aligned} &\int_t^{2t} e^{p\gamma s} \|(\phi_t E_t f)(\cdot, s)\|_X^p ds = \int_t^{2t} e^{p\gamma s} \phi_t(s)^p \|f(\cdot, 2t-s)\|_X^p ds \\ &\leq \int_{\max(0, t-1)}^t e^{2p\gamma(t-r)} e^{p\gamma r} \|f(\cdot, r)\|_X^p dr \leq e^{2p\gamma} \int_0^t e^{p\gamma r} \|f(\cdot, r)\|_X^p dr. \end{aligned}$$

Thus, noting that $\phi_t E_t f$ vanishes for $s \notin [0, 2t]$, we have

$$\|e^{\gamma s} \phi_t E_t f\|_{L_p(\mathbb{R}, X)} \leq e^{2\gamma} \|e^{\gamma s} f\|_{L_p((0,t), X)}. \quad (4.15)$$

Noting (3.4) and using (4.15), we have

$$\begin{aligned} &\|e^{\gamma s} (\phi_t E_t \mathbf{f}, \partial_s (\phi_t E_t \mathbf{g}))\|_{L_p(\mathbb{R}, L_q(\Omega))} \leq C e^{2\gamma_0} \|e^{\gamma s} (\mathbf{f}, \partial_t \mathbf{g})\|_{L_p((0,t), L_q(\Omega))}, \\ &\|e^{\gamma s} (\phi_t E_t g, \partial_s (\phi_t E_t \mathbf{h}))\|_{L_p(\mathbb{R}, W_q^1(\Omega))} \leq C e^{2\gamma_0} \|e^{\gamma s} (g, \mathbf{h})\|_{L_p((0,t), W_q^1(\Omega))} \end{aligned} \quad (4.16)$$

for any $\gamma \in (0, \gamma_0]$ with some constant independent of γ , t and T .

In addition, applying the same argumentation as in the proof of the inequality (3.2) in the appendix below, we have

$$\|e^{\gamma s} \tilde{\Lambda}_\gamma^{1/2} f\|_{L_p(\mathbb{R}, L_q(\Omega))} \leq C\{\|e^{\gamma s} \partial_s[(1 - \Delta)^{-1/2}(\iota f)]\|_{L_p(\mathbb{R}, L_q(\mathbb{R}^N))} + \|e^{\gamma s} f\|_{L_p(\mathbb{R}, W_q^1(\Omega))}\}, \quad (4.17)$$

so that using (4.15) and (3.4), we have

$$\begin{aligned} & \|e^{\gamma s}(\tilde{\Lambda}_\gamma^{1/2}(\phi_t E_t g), \tilde{\Lambda}_\gamma^{1/2}(\phi_t E_t \mathbf{h}))\|_{L_p(\mathbb{R}, L_q(\Omega))} \\ & \leq C e^{2\gamma_0} \{\|e^{\gamma s} \partial_s[(1 - \Delta)^{-1/2}(\iota g, \iota \mathbf{h})]\|_{L_p((0, t), L_q(\mathbb{R}^N))} + \|e^{\gamma s}(g, \mathbf{h})\|_{L_p((0, t), W_q^1(\Omega))}\}. \end{aligned} \quad (4.18)$$

Setting $\mathbf{u} = \mathbf{u}^T$ and $\theta = \theta^T$ and combining (4.14), (4.16) and (4.18), we have Theorem 4.5. \square

5 A proof of a global in time unique existence theorem

In this section, we prove Theorem 2.6, so that we assume that Ω is bounded in addition. Let T_0 be a positive number such that for any initial data $\mathbf{v}_0 \in \mathcal{D}_{q,p}(\Omega)$ with $\|\mathbf{v}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)} \leq 1$, problem (2.4) admits a unique solution $\mathbf{u} \in \mathcal{S}_{p,q}^{1,2}(0, T_0)$ satisfying (2.10). Here and in the following, we set

$$\begin{aligned} \mathcal{S}_{p,q}^{1,2}(a, b) &= W_p^1((a, b), L_q(\Omega)^N) \cap L_p((a, b), W_q^2(\Omega)^N), \\ \mathbb{I}_{\mathbf{v}}(a, b) &= \|e^{\gamma t} \partial_t \mathbf{v}\|_{L_p((a, b), L_q(\Omega))} + \|e^{\gamma t} \mathbf{v}\|_{L_p((a, b), W_q^2(\Omega))} \end{aligned}$$

for any a, b satisfying $0 \leq a < b \leq \infty$ for the notational simplicity, where γ is a fixed positive number for which Theorem 4.2 and Theorem 4.4 hold. By Theorem 2.4, such $T_0 > 0$ exists.

Let ϵ be a small positive number ≤ 1 that is determined later and we assume that $\mathbf{v}_0 \in \mathcal{D}_{q,p}(\Omega)$ and $\|\mathbf{v}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)} \leq \epsilon$. Let T be a positive number such that problem (2.4) admits a solution $\mathbf{u} \in \mathcal{S}_{p,q}^{1,2}(0, T)$ that satisfies (2.10). Since $\|\mathbf{v}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)} \leq \epsilon \leq 1$, we have $T \geq T_0$. The main step is to prove that there exist constants $\epsilon_0 > 0$ and M_6 independent of ϵ and T such that

$$\mathbb{I}_{\mathbf{v}}(0, t) \leq M_6(\epsilon + \mathbb{I}_{\mathbf{v}}(0, t)^2) \quad (5.1)$$

for any $t \in (0, T]$ provided that $0 < \epsilon \leq \epsilon_0$.

In fact, let $r_{\pm}(\epsilon)$ be two roots of the quadratic equation: $M_6(\epsilon + x^2) - x = 0$, that is $r_{\pm}(\epsilon) = (2M_6)^{-1} \pm \sqrt{(2M_6)^{-2} - \epsilon}$. We find a small positive number $\epsilon_1 > 0$ such that $0 < r_-(\epsilon) < r_+(\epsilon)$ whenever $0 < \epsilon \leq \epsilon_1$. In this case, $r_-(\epsilon) = M_6\epsilon + O(\epsilon^2)$ as $\epsilon \rightarrow 0$. Since $\mathbb{I}_{\mathbf{u}}(0, t) \rightarrow 0$ as $t \rightarrow 0$ and $\mathbb{I}_{\mathbf{u}}(0, t)$ is a continuous function with respect to t , by (5.1) we have $\mathbb{I}_{\mathbf{u}}(0, t) \leq r_-(\epsilon)$ for any $t \in (0, T]$, especially $\mathbb{I}_{\mathbf{u}}(0, T) \leq r_-(\epsilon)$. To prove

$$\sup_{t \in (0, T)} \|\mathbf{u}(\cdot, t)\|_{B_{q,p}^{2(1-1/p)}(\Omega)} \leq M_7(\mathbb{I}_{\mathbf{u}}(0, T) + e^{-\gamma T} \|\mathbf{v}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)}), \quad (5.2)$$

we take $\mathbf{Z} \in \mathcal{S}_{p,q}^{1,2}(0, \infty)$ which solves (3.18) with some pressure term Π . By Corollary 4.3 we have

$$\|e^{\gamma t} \partial_t \mathbf{Z}\|_{L_p(\mathbb{R}_+, L_q(\Omega))} + \|e^{\gamma t} \mathbf{Z}\|_{L_p(\mathbb{R}_+, W_q^2(\Omega))} \leq M_8 \|\mathbf{v}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)}$$

with some constant M_8 independent of ϵ and T , because \mathbf{v}_0 satisfies (2.14) when $\Gamma = \emptyset$. Employing the same argument as in the proof of (3.16), we have (5.2).

Since we may assume that $r_-(\epsilon) \leq 2M_0$, by (5.2) we have $\|\mathbf{u}(\cdot, T - 0)\|_{B_{q,p}^{2(1-1/p)}(\Omega)} \leq M_7(2M_0 + 1)\epsilon$, because $e^{-\gamma T} \leq 1$. Choose ϵ so small that $M_7(2M_0 + 1)\epsilon \leq 1$. By Theorem 2.4, there exists a unique solution $\mathbf{u}' \in \mathcal{S}_{p,q}^{1,2}(T, T + T_0)$ of the equations:

$$\begin{aligned} \partial_t \mathbf{u}' - \operatorname{Div} \mathbf{T}(\mathbf{u}', \theta') &= \mathbf{F}(\mathbf{u}'), \quad \operatorname{div} \mathbf{u}' = G(\mathbf{u}') = \operatorname{div} \mathbf{G}(\mathbf{u}') \quad \text{in } \Omega \times (T, T + T_0), \\ \mathbf{T}(\mathbf{u}', \theta') \mathbf{n}|_S &= \mathbf{H}(\mathbf{u}') \mathbf{n}|_S, \quad \mathbf{u}'|_{\Gamma} = 0, \quad \mathbf{u}'|_{t=T+0} = \mathbf{u}|_{t=T-0}, \end{aligned}$$

with some pressure term θ' . Choosing T_0 smaller if necessary, we may assume that

$$\int_T^{T+T_0} \|\nabla \mathbf{u}'(\cdot, t)\|_{L_\infty(\Omega)} dt \leq \sigma/2.$$

Since $\int_0^T \|\nabla \mathbf{u}(\cdot, t)\|_{L_\infty(\Omega)} dt \leq M_9 \mathbb{I}_{\mathbf{u}}(0, T)$ with some constant M_9 independent of ϵ and T as follows from (3.9), we choose ϵ so small that $M_9\epsilon < \sigma/2$, so that $\int_0^T \|\nabla \mathbf{u}(\cdot, t)\|_{L_\infty(\Omega)} dt \leq \sigma/2$. If we define \mathbf{u}'' by $\mathbf{u}''(\cdot, t) = \mathbf{u}(\cdot, t)$ for $0 \leq t \leq T$ and $\mathbf{u}''(\cdot, t) = \mathbf{u}'(\cdot, t)$ for $T \leq t \leq T + T_0$, then \mathbf{u}'' satisfies the equations (2.4) for $t \in (0, T + T_0)$ with some pressure term θ' and the condition: $\int_0^{T+T_0} \|\mathbf{u}''(\cdot, t)\|_{L_\infty(\Omega)} dt \leq \sigma$. Thus, by (5.1) $\mathbb{I}_{\mathbf{u}''}(0, T + T_0) \leq M_6(\epsilon + \mathbb{I}_{\mathbf{u}''}(0, T + T_0)^2)$. Repeating this argument, we can prolong \mathbf{u} to any time interval $(0, T)$ with $\mathbb{I}_{\mathbf{u}}(0, T) \leq r_-(\epsilon)$, which completes the existence of solution \mathbf{u} globally defined in time with $\mathbb{I}_{\mathbf{u}}(0, \infty) \leq r_-(\epsilon)$. The uniqueness follows from the same argumentations as in the proof of Theorem 2.4 with small $\epsilon > 0$ instead of small $T > 0$. Therefore, our task is to prove (5.1).

Applying Theorem 4.4 to problem (2.4), we have

$$\mathbb{I}_{\mathbf{u}}(0, t) \leq C\{\|\mathbf{v}_0\|_{B_{q,p}^{2(1-1/p)}(\Omega)} + \delta(\Gamma) \sum_{\ell=1}^M \left(\int_0^t |e^{\gamma s}(\mathbf{u}(\cdot, s), \mathbf{p}_\ell)_\Omega|^p ds \right)^{1/p} + \mathbb{K}_{\mathbf{u}}(0, t)\} \quad (5.3)$$

for any $t \in (0, T]$ with

$$\begin{aligned} \mathbb{K}_{\mathbf{u}}(0, t) &= \|e^{\gamma s}(\mathbf{F}(\mathbf{u}), \partial_s \mathbf{G}(\mathbf{u}))\|_{L_p((0,t), L_q(\Omega))} + \|e^{\gamma s}(G(\mathbf{u}), \mathbf{H}(\mathbf{u})\tilde{\mathbf{n}})\|_{L_p((0,t), W_q^1(\Omega))} \\ &\quad + \|e^{\gamma s} \partial_s [(1 - \Delta)^{-1/2}(\iota G(\mathbf{u}), \iota \mathbf{H}(\mathbf{u})\tilde{\mathbf{n}})]\|_{L_p((0,t), L_q(\mathbb{R}^N))}. \end{aligned}$$

Here and in the following, C denotes a generic constant independent of ϵ , $t \in (0, T]$ and T .

When $\Gamma = \emptyset$, $\delta(\Gamma) = 1$, so that we have to estimate $\int_0^t |e^{\gamma t}(\mathbf{u}(\cdot, s), \mathbf{p}_\ell)_\Omega|^p ds$. Recalling Remark 2.5 (1), $\mathbf{v}(x, t) = \mathbf{u}(\mathbf{X}_{\mathbf{u}}^{-1}(x, t), t)$ satisfies the equation (1.1) with (1.2), where $\mathbf{X}^{-1}(x, t)$ denotes the inverse map of the correspondence: $x = \xi + \int_0^t \mathbf{u}(\xi, s) ds = \mathbf{X}_{\mathbf{u}}(\xi, t)$. Since $\frac{d}{dt} \int_{\Omega_t} \mathbf{v}(x, t) \mathbf{p}_\ell(x) dx = 0$, by (2.14) we have $\int_{\Omega_t} \mathbf{v}(x, t) \mathbf{p}_\ell(x) dx = 0$, so that

$$\int_{\Omega} \mathbf{u}(\xi, s) \mathbf{p}_\ell(\xi + \int_0^s \mathbf{u}(\xi, r) dr) d\xi = 0,$$

which combined with (3.9) and Hölder's inequality furnishes that

$$\begin{aligned} |(\mathbf{u}(\cdot, s), \mathbf{p}_\ell)_\Omega| &\leq C \|\mathbf{u}(\cdot, s)\|_{L_q(\Omega)} \int_0^s \|\mathbf{u}(\cdot, r)\|_{W_q^1(\Omega)} dr \\ &\leq C \|\mathbf{u}(\cdot, s)\|_{L_q(\Omega)} \left(\int_0^s e^{-p'\gamma r} ds \right)^{1/p'} \left(\int_0^t (e^{\gamma r} \|\mathbf{u}(\cdot, r)\|_{W_q^1(\Omega)})^p dr \right)^{1/p}. \end{aligned}$$

Thus, we have

$$\delta(\Gamma) \sum_{\ell=1}^M \left(\int_0^t |e^{\gamma s}(\mathbf{u}(\cdot, s), \mathbf{p}_\ell)_\Omega|^p ds \right)^{1/p} \leq C \mathbb{I}_{\mathbf{u}}(0, t)^2. \quad (5.4)$$

From now on, we estimate $\mathbb{K}_{\mathbf{u}}(0, t)$. By Hölder's inequality we have

$$\sup_{s \in (0, t)} \left\| \int_0^s \nabla \mathbf{u}(\cdot, r) dr \right\|_{W_q^1(\Omega)} \leq \left(\int_0^s (e^{\gamma r} \|\mathbf{u}(\cdot, r)\|_{W_q^2(\Omega)})^p dr \right)^{1/p} \left(\int_0^s e^{-\gamma r p'} dr \right)^{1/p'} \leq C \mathbb{I}_{\mathbf{u}}(0, t). \quad (5.5)$$

Since (2.10) holds and since we may assume that $\sigma \leq 1$, by (2.6), (5.5) and (3.9)

$$\begin{aligned} \sup_{s \in (0, t)} \left\| \int_0^s \nabla \mathbf{u}(\cdot, r) dr \right\|_{L_\infty(\Omega)} &\leq 1, \quad \sup_{s \in (0, t)} \|\mathbf{V}_i(\int_0^s \nabla \mathbf{u}(\cdot, r) dr)\|_{W_q^1(\Omega)} \leq C \mathbb{I}_{\mathbf{u}}(0, t), \\ \sup_{s \in (0, t)} \|\nabla \mathbf{W}(\int_0^s \nabla \mathbf{u}(\cdot, r) dr)\|_{L_q(\Omega)} &\leq C \mathbb{I}_{\mathbf{u}}(0, t), \end{aligned} \quad (5.6)$$

where $i = 1, 2, 4, 5$ and 6 and $\mathbf{W} = \mathbf{W}(\mathbf{K})$ is any matrix of polynomials with respect to \mathbf{K} . By (3.9), (1.7), (2.5) and (5.6), we have

$$\|\mathbf{F}(\mathbf{u})\|_{L_p((0,t),L_q(\omega))} \leq C\mathbb{I}_{\mathbf{u}}(0,t)^2, \quad \|(G(\mathbf{u}), \mathbf{H}(\mathbf{u})\tilde{\mathbf{n}})\|_{L_p((0,t),W_q^1(\Omega))} \leq C\mathbb{I}_{\mathbf{u}}(0,t)^2. \quad (5.7)$$

Since $B_{q,p}^{2(1-1/p)}(\Omega) \subset W_q^1(\Omega)$ as follows from the assumption: $2 < p, \infty$, by (5.2) we have

$$\sup_{s \in (0,t)} \|\mathbf{u}(s, \cdot)\|_{W_q^1(\Omega)} \leq C(\mathbb{I}_{\mathbf{u}}(0,t) + \epsilon), \quad \sup_{s \in (0,t)} \|\partial_s \mathbf{W}(\int_0^s \nabla \mathbf{u}(\cdot, r) dr)\|_{L_q(\Omega)} \leq C(\mathbb{I}_{\mathbf{u}}(0,t) + \epsilon). \quad (5.8)$$

Thus, by (3.9), (5.6) and (5.8)

$$\|\partial_s \mathbf{G}(\mathbf{u})\|_{L_p((0,t),L_q(\Omega))} \leq C(\mathbb{I}_{\mathbf{u}}(0,t)^2 + (\mathbb{I}_{\mathbf{u}}(0,T) + \epsilon)\mathbb{I}_{\mathbf{u}}(0,T)) \leq 2C(\mathbb{I}_{\mathbf{u}}(\mathbf{u})^2 + \epsilon), \quad (5.9)$$

because $0 < \epsilon \leq 1$.

To estimate $\|e^{\gamma t}(\tilde{\Lambda}_\gamma^{1/2} g_{\mathbf{u}}, \tilde{\Lambda}_\gamma^{1/2}(\mathbf{h}_{\mathbf{u}}\tilde{\mathbf{n}}))\|_{L_p(\mathbb{R},L_q(\Omega))}$, we use the following lemma which can be proved in the same manner as in the proof of Lemma 3.3.

Lemma 5.1. *Let $1 < p, q < \infty$, $N < r, q < \infty$ and let Ω be a uniform $W_r^{2-1/r}$. Let ι be the extension map satisfying the properties (e-1) and (e-2). Then*

$$\begin{aligned} & \|e^{\gamma s} \tilde{\Lambda}_\gamma^{1/2}((\nabla f)g)\|_{L_p((0,t),L_q(\Omega))}^p \\ & \leq C \left\{ \int_0^t (e^{-\gamma s} \|\partial_s f(\cdot, s)\|_{L_q(\Omega)} \|g(\cdot, s)\|_{W_q^1(\Omega)})^p ds + \int_0^t (e^{\gamma s} \|\nabla f(\cdot, s)\|_{L_q(\Omega)} \|\partial_t g(\cdot, s)\|_{L_q(\Omega)})^p ds \right\}. \end{aligned}$$

Applying Lemma 5.1 and using (1.7), (5.6) and (5.7), we have

$$\begin{aligned} & \|e^{\gamma s} \partial_s [(1 - \Delta)^{-1/2} (\iota G(\mathbf{u}), \iota \mathbf{H}(\mathbf{u})\tilde{\mathbf{n}})]\|_{L_p((0,t),L_q(\mathbb{R}^N))} \\ & \leq C(\mathbb{I}_{\mathbf{u}}(0,t)^2 + (\mathbb{I}_{\mathbf{u}}(0,T) + \epsilon)\mathbb{I}_{\mathbf{u}}(0,T)) \leq 2(\mathbb{I}_{\mathbf{u}}(0,T)^2 + \epsilon), \end{aligned} \quad (5.10)$$

which combined with (5.3), (5.4), (5.7) and (5.9) furnishes (5.1). This completes the proof of Theorem 2.6.

A A proof of the inequality (3.2)

First, we prove the inequality (3.2) in case of $\Omega = \mathbb{R}^N$.

Lemma A.1. *Let $1 < p, q < \infty$ and set $(1 - \Delta)^s f - \mathcal{F}_\xi^{-1}[(1 + |\xi|^2)^{s/2} \hat{f}(\xi)]$ for $f \in \mathcal{S}'(\mathbb{R}^N)$ and $s \in \mathbb{R}$. Here, \hat{f} denotes the Fourier transform of f , \mathcal{F}_ξ^{-1} the inverse Fourier transform and $\mathcal{S}'(\mathbb{R}^N)$ the space of tempered distributions on \mathbb{R}^N in the sense of L. Schwartz. Then, we have*

$$\begin{aligned} & \|e^{-\gamma t} \Lambda_\gamma^{1/2} f\|_{L_p(\mathbb{R},L_q(\mathbb{R}^N))} \\ & \leq C \{ \|e^{-\gamma t} f\|_{L_p(\mathbb{R},L_q(\mathbb{R}^N))} + \|e^{-\gamma t} (1 - \Delta)^{-1/2} \partial_t f\|_{L_p(\mathbb{R},L_q(\mathbb{R}^N))} + \|e^{-\gamma t} (1 - \Delta)^{1/2} f\|_{L_p(\mathbb{R},L_q(\mathbb{R}^N))} \}. \end{aligned}$$

Proof. The idea of our proof here is the same as in the proof of Proposition 2.8 in [24]. Let $\varphi_0(t)$ be a function in $C^\infty(\mathbb{R})$ such that $\varphi_0(t) = 1$ for $|t| \leq 1$ and $\varphi_0(t) = 0$ for $|t| \geq 2$, and set $\varphi_\infty(t) = 1 - \varphi_0(t)$. We define functions $A_j(\xi, \lambda)$ ($j = 1, 2$) by

$$\begin{aligned} A_1(\xi, \lambda) &= \varphi_\infty(\tau) \varphi_0\left(\frac{(1 + |\xi|^2)^{1/2}}{|\lambda|}\right) \frac{(1 + |\xi|^2)^{1/4} \lambda^{1/2}}{\lambda}, \\ A_2(\xi, \lambda) &= \varphi_\infty(\tau) \varphi_\infty\left(\frac{(1 + |\xi|^2)^{1/2}}{|\lambda|}\right) \frac{\lambda^{1/2}}{(1 + |\xi|^2)^{1/4}}. \end{aligned}$$

We have

$$|\partial_\tau^\ell \partial_\xi^\alpha A_j(\xi, \lambda)| \leq C_{\ell,\alpha} |\tau|^{-\ell} |\xi|^{-|\alpha|} \quad (\lambda = i\tau + \gamma, \quad j = 1, 2)$$

for any $\ell \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^N$, $\xi \in \mathbb{R}^N \setminus \{0\}$, $\tau, \gamma \in \mathbb{R} \setminus \{0\}$ with some constant $C_{\ell, \alpha}$ depending solely on ℓ and α . Set $A_j(\lambda, D_x)f = \mathcal{F}_\xi^{-1}[A_j(\xi, \lambda)\hat{f}(\xi)]$ for any $f \in \mathcal{S}'(\mathbb{R}^N)$, and then by Theorem 3.3 in [12] we know that the sets $\{\tau^k \partial_\tau^k A_j(\lambda, D_x) \mid \tau \in \mathbb{R} \setminus \{0\}\}$ are \mathcal{R} -bounded families in $\mathcal{L}(L_q(\mathbb{R}^N))$ and their \mathcal{R} -bounded are less than $C_{q, N} \max_{|\alpha| \leq N+2} C_{k, \alpha}$ for $k = 0, 1$ and $j = 1, 2$, where $\mathcal{L}(L_q(\mathbb{R}^N))$ is the set of all bounded linear operators on $L_q(\mathbb{R}^N)$. Therefore, by Weis's operator valued Fourier multiplier theorem [38] we have

$$\|e^{-\gamma t} A_j(\partial_t, D_x)F\|_{L_p(\mathbb{R}, L_q(\mathbb{R}^N))} \leq C \|e^{-\gamma t} F\|_{L_p(\mathbb{R}, L_q(\Omega))} \quad (\gamma \neq 0). \quad (\text{A.1})$$

Here, the operators $A_j(\partial_t, D_x)$ are defined by

$$A_j(\partial_t, D_x)F = \mathcal{L}_\lambda^{-1}[A_j(\lambda, D_x)\mathcal{L}[F](\lambda, \cdot)](t).$$

Dividing $\lambda^{1/2}$ into the following three parts:

$$\lambda^{1/2} = \varphi_0(\tau)\lambda^{1/2} + \varphi_\infty(\tau)A_1(\xi, \lambda)\frac{\lambda}{(1+|\xi|^2)^{1/4}} + \varphi_\infty(\tau)A_2(\xi, \lambda)(1+|\xi|^2)^{1/4},$$

and using (A.1) and Bourgain's Fourier multiplier theorem [8], we have Lemma A.1. \square

Proof of the inequality (3.2). To prove the lemma, we use the extension map E having the following properties:

$$(\text{Ex-1}) \quad \|Ef\|_{L_q(\mathbb{R}^N)} \leq C_{q, \Omega} \|f\|_{L_q(\Omega)}$$

$$(\text{Ex-2}) \quad \|(1 - \Delta)^{1/2} Ef\|_{L_q(\mathbb{R}^N)} \leq C_{q, \Omega} \|f\|_{W_q^1(\Omega)}$$

$$(\text{Ex-3}) \quad \|(1 - \Delta)^{-1/2} E(\nabla f)\|_{L_q(\mathbb{R}^N)} \leq C_{q, \Omega} \|f\|_{L_q(\Omega)}.$$

Such extension map can be constructed under the assumption that $N < q, r < \infty$. By Lemma A.1, we have

$$\begin{aligned} \|e^{-\gamma t} \Lambda_\gamma^{1/2}((\nabla f)g)\|_{L_p(\mathbb{R}, L_q(\Omega))} &\leq \|e^{-\gamma t} \Lambda_\gamma^{1/2} E((\nabla f)g)\|_{L_p(\mathbb{R}, L_q(\mathbb{R}^N))} \\ &\leq \|e^{-\gamma t} E((\nabla f)g)\|_{L_p(\mathbb{R}, L_q(\Omega))} + \|e^{-\gamma t} (1 - \Delta)^{-1/2} \partial_t E((\nabla f)g)\|_{L_p(\mathbb{R}, L_q(\Omega))} \\ &\quad + \|e^{-\gamma t} (1 - \Delta)^{1/2} ((\nabla f)g)\|_{L_p(\mathbb{R}, L_q(\Omega))}. \end{aligned}$$

Using the identity: $\partial_t((\nabla f)g) = \nabla(\partial_t f \cdot g) - \partial_t f(\nabla g) + (\nabla f)\partial_t g$ and (Ex-3), we have

$$\begin{aligned} \|e^{-\gamma t} (1 - \Delta)^{-1/2} \partial_t E((\nabla f)g)\|_{L_p(\mathbb{R}, L_q(\Omega))} &\leq C \|e^{-\gamma t} (\partial_t f \cdot g)\|_{L_p(\mathbb{R}, L_q(\Omega))} \\ &\quad + \|e^{-\gamma t} (1 - \Delta)^{-1/2} E(\partial_t f(\nabla g))\|_{L_p(\mathbb{R}, L_q(\mathbb{R}^N))} + \|e^{-\gamma t} (1 - \Delta)^{-1/2} E((\nabla f)\partial_t g)\|_{L_p(\mathbb{R}, L_q(\mathbb{R}^N))}. \end{aligned}$$

By (3.9) and (Ex-1), we have

$$\begin{aligned} \|e^{-\gamma t} (\partial_t f \cdot g)\|_{L_p(\mathbb{R}, L_q(\Omega))}^p &\leq C \int_{-\infty}^{\infty} (e^{-\gamma t} \|\partial_t f(\cdot, t)\|_{L_q(\Omega)} \|g(\cdot, t)\|_{W_q^1(\Omega)})^p dt, \\ \|e^{-\gamma t} E((\nabla f)g)\|_{L_p(\mathbb{R}, L_q(\Omega))}^p &\leq C \int_{-\infty}^{\infty} (e^{-\gamma t} \|\nabla f(\cdot, t)\|_{L_q(\Omega)} \|g(\cdot, t)\|_{W_q^1(\Omega)})^p dt. \end{aligned}$$

To estimate other terms, we use the inequality:

$$\|(1 - \Delta)^{-1/2} E(fg)\|_{L_q(\mathbb{R}^N)} \leq C \|f\|_{L_q(\Omega)} \|g\|_{L_q(\Omega)}. \quad (\text{A.2})$$

In fact, for any $\varphi \in C_0^\infty(\mathbb{R}^N)$, we observe that

$$\begin{aligned} |((1 - \Delta)^{-1/2} E(fg), \varphi)_{\mathbb{R}^N}| &\leq \|E(fg)\|_{L_{q/2}(\mathbb{R}^N)} \|(1 - \Delta)^{-1/2} \varphi\|_{L_s(\mathbb{R}^N)} \\ &\leq C \|f\|_{L_q(\Omega)} \|g\|_{L_q(\Omega)} \|(1 - \Delta)^{-1/2} \varphi\|_{L_s(\mathbb{R}^N)}, \end{aligned}$$

where s is an index such that $1/s + 2/q = 1$. Since $2 \leq N < q < \infty$, we can choose such s with $1 < s < \infty$. Since $N(1/q' - 1/s) = N(1 - 1/q - 1/s) = N/q < 1$, we have $\|(1 - \Delta)^{-1/2}\varphi\|_{L_s(\mathbb{R}^N)} \leq C\|(1 - \Delta)^{-1/2}\varphi\|_{W_{q'}^1(\mathbb{R}^N)}$, so that we have (A.2).

By (A.2) we have

$$\begin{aligned} \|e^{-\gamma t}(1 - \Delta)^{-1/2}E(\partial_t f(\nabla g))\|_{L_p(\mathbb{R}, L_q(\mathbb{R}^N))}^p &\leq C \int_{-\infty}^{\infty} (e^{-\gamma t} \|\partial_t f(\cdot, t)\|_{L_q(\Omega)} \|\nabla g(\cdot, t)\|_{L_q(\Omega)})^p dt, \\ \|e^{-\gamma t}(1 - \Delta)^{-1/2}E((\nabla f)\partial_t g)\|_{L_p(\mathbb{R}, L_q(\mathbb{R}^N))}^p &\leq C \int_{-\infty}^{\infty} (e^{-\gamma t} \|\nabla f(\cdot, t)\|_{L_q(\Omega)} \|\partial_t g(\cdot, t)\|_{L_q(\Omega)})^p dt. \end{aligned}$$

By (3.9) and (Ex-2) we have

$$\begin{aligned} \|e^{-\gamma t}(1 - \Delta)^{1/2}E((\nabla f)g)\|_{L_p(\mathbb{R}, L_q(\mathbb{R}^N))}^p &\leq C \|e^{-\gamma t}((\nabla f)g)\|_{L_p(\mathbb{R}, W_q^1(\mathbb{R}^N))}^p \\ &\leq C \int_{-\infty}^{\infty} (e^{-\gamma t} \|\nabla f(\cdot, t)\|_{W_q^1(\Omega)} \|\nabla g(\cdot, t)\|_{W_q^1(\Omega)})^p dt. \end{aligned}$$

This completes the proof of the inequality (3.2).

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